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ELLIPTIC INTEGRALS.

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ELLIPTIC INTEGRALS.

BY

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Large Modulus c.

29. We now transform $C \,.\, G(q, x) = \frac{d}{dx} \,.\, \log \Theta(q, x)$, and obtain

$$B \,.\, J(c\omega) = 1 - \sqcap(ax) + (\sqcap \,.\, a \,.\, \overline{\pi - x} - \sqcap \,.\, a \,.\, \overline{\pi + x})$$
$$+ (\sqcap a \,.\, \overline{2\pi - x} - \sqcap a \,.\, \overline{2\pi + x})$$
$$+ (\sqcap a \,.\, \overline{3\pi - x} - \sqcap a \,.\, \overline{3\pi + x}) + \&c.,$$

where $\sqcap$ stands for the Anticyclic $(1 - \text{Tan})$.

The last evades the *fraction* of Art. 26, but the multiplier $a = \frac{C}{B}$ probably outweighs the advantage.

CHAPTER IX.

The Higher Scales.

1. From $c \sin \omega = \frac{\Lambda(q, x)}{\Theta(q, x)}$, we imitate $h \sin \theta = \frac{\Lambda(q^n, nx)}{\Theta(q^n, nx)}$ for a new scale whose index shall be n, which we have supposed to be a positive integer, and in general an *odd* number.

With appropriate arcs $\omega' \omega'' \ldots \omega^{(n-1)}$, we obtain

$$\sqrt{h} \sin \theta = \sqrt{c} \sin \omega \,.\, \sqrt{c} \,.\, \sin \omega' \,.\, \sqrt{c} \sin \omega'' \ldots \sqrt{c} \,.\, \sin \omega^{(n-1)}.$$

Also $$F(c\omega) = \mu \,.\, F(h\theta).$$

2. Thus we transform to $\sqrt{h} \sin \theta = \sqrt{c^n} \,.\, \sin \omega \,.\, \mathfrak{P} \,.\, \frac{\sin^2 a_{2p} - \sin^2 \omega}{1 - c^2 \sin^2 a_{2p} \,.\, \sin^2 \omega}$ up to $p = \frac{n-1}{2}$.

3. To find $\cos \theta$ is now a problem of easy algebra, viz.

$$\cos \theta \,.\, \mathfrak{P}(1 - c^2 \sin^2 a_{2p} \,.\, \sin^2 \omega) = \cos \omega_0 \, \mathfrak{P}\left(1 - \frac{\sin^2 \omega}{\sin^2 a_{2p-1}}\right).$$

4. By a lawful *double inversion* we next deduce $\Delta(h, \theta)$

$$\frac{\Delta(h\theta)}{\Delta(c\omega)} = \mathfrak{P} \, \frac{1 - c^2 \sin^2 \omega_{2p-1}}{1 - c^2 \sin^2 a_{2p}}.$$

5. By integrating the last, we obtain $\tan \frac{1}{2}(\theta - \omega) = \Sigma \tan^{-1} \,.\, \{\Delta(c, a_{2p}) \tan \omega\}$.

6. Jacobi's second Theorem is obtained by making

$$\sin \omega = \sqrt{-1} \,.\, \tan \psi, \quad \sin \theta = \sqrt{-1} \,.\, \tan \chi.$$

7, 8. Proof that

$$CG(c\omega) - HG(h, \theta) = C \,.\, \Sigma M \,.\, \frac{2 \sin^2 \beta \sin \omega \cos \omega \, \Delta(c\omega)}{1 - c^2 \sin^2 \beta \,.\, \sin^2 \omega},$$

where β means 2, 4, 6... the *even* indices of $(n-1)$, and M is a series depending on β.

9—11. Proof that for every index β the $M = \frac{1}{n}$.

Finally $\daleth(c\omega) = \frac{1}{n} \daleth(h, \theta) - \frac{1}{n} \Sigma \log(1 - c^2 \sin^2 \beta \sin^2 \omega)$.

12. Values of $\aleph_0 - b^2 - \frac{1}{n\mu^2}(\aleph_\lambda - g^2)$ and of $1 - \aleph_q = \frac{n}{\mu^2}(1 - \aleph_\lambda)$.

PART III. ON THE PARANOME.

CHAPTER X.

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CHAPTER XI.

Paranome with Three Irreducible Elements.

PRELIMINARY NOTICE.

As a virtual introduction to the present topic, a short treatise on Anticyclics has already been issued from Messrs Macmillan and Bowes, with Skeleton tables. Much ampler tables remain in my closet; a fact which indicates how long I have been planning the present attempt. Indeed in 1852 the substance of Chapters X. and XI. below appeared in the *Dublin and Cambridge Mathematical Journal*, my aim being to reduce every function that needed to be found by infinite series to a form that vanished at every quadrant; just as Legendre's $\Upsilon(c\omega) = \int_0 E(c\omega)\, d\,.\,F$ is superseded by my $\daleth(c\omega)$. I had already completed in outline the present treatise, except Chapter VI.

So much I recite, lest I be thought to place myself in rivalry to Professor Cayley, who doubtless has views far larger than mine, and adapts his notation to researches which I neither touch nor imagine. Only quite late in time was I aware that he had written a treatise on these Integrals: my earlier pages were already in the printer's hands. My preoccupations preclude my study of it: indeed, years ago I found that to compare Mr Russell's equations in the British Association with my own was similar to translating from a new language. I comfort myself by thinking that the lines of argument and the notation which is easiest to *me* may help some students who cannot give their chief energies to mathematics.

CORRIGENDA.

Page 110, line 14. Read $\frac{F(c_1\eta)}{C_1} = x + y = \frac{F(c\omega)}{C} + \frac{F(c\theta)}{C}$, and in the corresponding denominators of the three following Corollaries change c, c_1 to C, C_1

Page 136, line 11. For r in left member write r_1

Page 139. In eq. (5), right member, the exponent ought to be $(n+u)^2$

Page 193, line 5 from foot. Read $=\sqrt{TP}\,(c\omega p) + \sqrt{TP}\,(c,\ \omega,\ -r)$,

Page 193, line 4 from foot. For Ω_3 (twice) read $\overset{3}{\Omega}$

Page 194, line 1. Read

Eliminate $\overset{3}{\Omega}$, $\quad .. \Omega - \frac{1}{2}\Omega_1 = \frac{1}{2}\tan^{-1}(\sqrt{p_1}\,.\,\sin\omega_1)$,

Page 200, line 10 from foot. For b_1 read b,

ELLIPTIC INTEGRALS.

LOWER THEORY.

CHAPTER I.

ELEMENTARY TREATMENT.

Notation and Definitions.

1. LEGENDRE is here followed in all main principles, as easiest to students who do not aspire to Mathematics as the professional pursuit of their lives.

In this theory Δ is not used to denote an *increment*, but as the *Cardinal Surd* of our integral. Thus $\Delta(c, \omega)$ stands for

$$\sqrt{(1 - c^2 \sin^2 \omega)},$$

and where conciseness is urgent, may be abridged into $\Delta(\omega)$ or into simple Δ, if c and ω can be *understood*.

The series for Δ^{-1} by the Binomial Theorem is

$$1 + \tfrac{1}{2}c^2 \sin^2 \omega + \frac{1.3}{2.4} . c^4 \sin^4 \omega + \&c.$$

It is worth while to write $k_1 k_2 k_3 \ldots$ for these numerical coefficients, so that k_n means $\dfrac{1.3.5 \ldots 2n-1}{2.4.6 \ldots 2n}$, at least in our early chapters.

2. By systematic Reduction which is not here needed, Legendre shows, that if R denotes a rational function,

$$\int R\,(x\sqrt{a+bx+cx^2+ex^3+fx^4})\,dx$$

can be expressed in the form of rational functions of x, with logarithmic and circular functions of x, by the aid of only *three* new forms, which are called ELLIPTIC (for a reason truly insufficient), and are of the types,

$$\int_0 \frac{d\omega}{\Delta},\quad \int_0 \Delta d\omega,\ \text{and}\ \int_0 \frac{d\omega}{(1+p)\sin\omega\,.\,\Delta};$$

if Δ mean $\sqrt{(1-c^2\sin^2\omega)}$.

Legendre proposed to call the three forms the Nome, the Epinome and the Paranome. The names are convenient, and no one had a better right to give them than he.

3. In detail, the constant c is called the *Modulus*, and the variable arc ω, the *Amplitude:* not but that the Modulus may vary independently, or indeed with p the additional *constant* in the third integral. This p is called the *Parameter*, and gives the name to the *Para*-nome.

Also b is a constant defined by the equation $b^2+c^2=1$, and is here named the *sub*modulus. Legendre has an arc θ, making $\sin\theta=c$ and $\cos\theta=b$: but I reserve θ, in close analogy with Legendre, and prefer an arc γ, such that $\sin\gamma=c$, $\cos\gamma=b$. I call γ the *Grade*, F is the symbol of the Nome, E of the Epinome, Π of the Paranome; or, with the elements, $F(c,\ \omega)$, $E\,(c,\ \omega)$, $\Pi\,(c,\ \omega,\ p)$. But the treatment of Π must be delayed until that of F and E is complete. In the original reduction p must have the form $\alpha+\beta\sqrt{-1}$ as its widest sense.

Legendre's Table of Reduction.

4. These equations are verified by mere Differentiation.

$$\int_0 \frac{\sin^2\omega}{\Delta}\,d\omega \quad \frac{F-E}{c^2}$$

$$\int_0 \frac{\cos^2\omega}{\Delta}\,d\omega = \frac{E-b^2F}{c^2}$$

$$\int_0 \frac{d\omega}{\Delta^3} = \frac{E}{b^2} - \frac{c^2\sin\omega\cos\omega}{b^2\,.\,\Delta}$$

$$\int_0 \frac{\cos^2\omega}{\Delta^3}\,.\,d\omega = \frac{F-E}{c^2} - \frac{\sin\omega\cos\omega}{\Delta}$$

$$\int_0 \frac{\sin^2\omega}{\Delta^3}\,.\,d\omega = \frac{E-b^2F}{b^2c^2} - \frac{\sin\omega\cos\omega}{b^2\Delta}$$

$$\int_0 \frac{\Delta d\omega}{\cos^2\omega} = \Delta\cos\omega + F - E$$

$$\int_0 \tan^2\omega\,.\,\Delta d\omega = \Delta\tan\omega + F - 2E$$

$$\int_0 \frac{\tan^2\omega}{\Delta}\,d\omega = \frac{\Delta\tan\omega - E}{b^2}$$

$$\int \frac{-\cot^2\omega}{\Delta}\,d\omega = \Delta\cot\omega + E$$

$$\int_0 \frac{1-\cos\omega}{1+\cos\omega}\,.\,\frac{d\omega}{\Delta} = 2\Delta\tan\frac{\omega}{2} + F - 2E$$

$$\int_0 3\Delta^3\,.\,d\omega = c^2\Delta\sin\omega\cos\omega + 2(1+b^2E) + b^2F$$

$$\int_0 3c^2\cos^2\omega\,\Delta d\omega = \Delta\sin\omega\cos\omega + (1+c^2)E - b^2F$$

$$\int_0 \frac{3b^2}{\Delta^5}\,.\,d\omega = 2(1+b^2)E - F - \frac{c^2}{b^2}\,.\,\frac{\sin\omega\cos\omega}{\Delta}\,.\left(2+2b^2+\frac{b^3}{\Delta^2}\right)$$

$\left[\int \frac{dF}{\Delta^{2r}},\quad \int_0 \frac{dF}{(\cos\omega)^{2r}},\quad \int \frac{dF}{(\sin\omega)^{2r}}\right.$ are all reducible to F and $\left.E\right]$

$$\Pi(c,\ \omega,\ -c^2) = \int_0 \frac{d\omega}{\Delta^3}$$

$$\Pi(c,\ \omega,\ -1) = \int_0 \frac{d\omega}{\Delta\,(\cos\omega)^2}$$

Periodicity of the Integrals.

5. If p is negative, and $1+p$ less than 1, there is a value of ω at which $1+p\sin^2\omega$ vanishes, and Π becomes infinite. Except in this case, ω may begin from zero, and increase indefinitely; then

$$\Delta^{-1},\ \Delta,\ (\overline{1+p}\,.\,\Delta)^{-1}$$

remaining positive, the three integrals increase perpetually with ω.

To include all three, for a moment write $\phi(\omega)$ for

$$\int_0 \frac{1-m\sin^2\omega}{1+p\sin^2\omega}\,.\,\frac{d\omega}{\Delta},$$

then if $m=-p$, $\phi(\omega)=F(\omega)$; if $p=0$ and $m=c^2$, $\phi(\omega)=E(\omega)$; and if $m=0$, $\phi(\omega)=\Pi(\omega)$.

In all cases, $\phi(-\omega) = -\phi(\omega)$, therefore each of the three functions is odd, each begins from zero.

If n is integer, (π as usual in Trigonometry), $\sin^2(n\pi + \omega) = \sin^2\omega$, also $d(n\pi + \omega) = d\omega$, hence $d.\phi(n\pi + \omega) = d\phi(\omega)$. Integrated, it is, $\phi(n\pi + \omega) = \phi(n\pi) + \phi(\omega)$ in which ω may be changed to $-\omega$, or $\phi(n\pi - \omega) = \phi(n\pi) - \phi(\omega)$; whence we easily deduce $\phi(n\pi) = n.\phi(\pi)$. Again, since $\sin(\pi - \omega) = \sin\omega$, $d(\pi - \omega) = -d\omega$,

$$\therefore\ \phi(\pi - \omega) = \text{const.} - \phi(\omega) = \phi(\pi) - \phi(\omega).$$

Make $\omega = \frac{1}{2}\pi$, then $\phi(\pi) = 2.\phi(\frac{1}{2}\pi)$. Finally

$$\phi(n\pi + \omega) = 2n.\phi(\tfrac{1}{2}\pi) + \phi(\omega).$$

The integral $\phi(\frac{1}{2}\pi)$ is called complete. In English custom F_c E_c represent $F(c, \frac{1}{2}\pi)$, $E(c, \frac{1}{2}\pi)$.

N.B. In these pages the capitals C, B *are not equivalent* to my E_c, F_b (as in Legendre's Supplements), but $F_c = \frac{1}{2}\pi.C$, $F_b = \frac{1}{2}\pi.B$. Knowing F and E from $\omega = 0$ to $\omega = \frac{1}{2}\pi$, we know them wholly.

To know the *ratio* of E_c to F_c seems generally the most convenient way of knowing E_c. I propose to call this ratio the *Ancilla*, and denote it by the Hebrew Alpha, $\therefore\ \aleph_c = \dfrac{E_c}{F_c}$. Beneath $\aleph$ is placed its appropriate *modulus*.

Geometrical Representation of F.

6. In elementary treatises it is shown that our $E(c, \omega)$ is the arc of an *Ellipse*, beginning from $\omega = 0$ at the extremity of the minor axis; but of F there is no obvious picture. I have thought that it is most easily imagined by a curve in *space*. Take an upright cylinder standing on a circular base whose radius is 1. Let ω be an arc of this circle, and from its variable extremity erect z, a generatrix of the cylinder, of a length so related to ω, that the spiral which its end marks on the cylinder shall *everywhere* be of the length $F(c\omega)$. This requires

$$(dz)^2 + (d\omega)^2 = (dF)^2 = \frac{d\omega^2}{1 - c^2\sin^2\omega};$$

or

$$dz^2 = \frac{c^2\sin^2\omega\, d\omega^2}{1 - c^2\sin^2\omega};$$

that is

$$dz = \frac{c\sin\omega\, d\omega}{\sqrt{(1 - c^2\sin^2\omega)}}.$$

To integrate this, put $c\cos\omega = v$,

$$\therefore\ 1 - c^2\sin^2\omega = 1 - (c^2 - v^2) = b^2 + v^2,$$

or
$$dz = \frac{-dv}{\sqrt{(b^2+v^2)}}.$$

Integrated,
$$z = \log\frac{1+c}{\sqrt{(b^2+v^2)}+v},$$

since when $z = 0$, $\omega = 0$ and $v = c\cos\omega = c$. Restoring Δ for

$$\sqrt{(b^2+v^2)},$$

we have
$$z = \log\frac{1+c}{\Delta + c\cos\omega},$$

$$(1+c)\,\epsilon^{-z} = \Delta + c\cos\omega.$$

This coexists with
$$(1-c)\ \epsilon^{z} = \Delta - c\cos\omega.$$

The spiral, however far continued, will represent $F(c\omega)$ and when $\omega = \frac{1}{2}\pi$, $z = \log\dfrac{1+c}{1-c}$.

Extreme values of the Modulus.

7. When c is so small that c^2 is too minute to notice, so is $c^2\sin^2\omega$; $\therefore\ \Delta = 1$. This confounds $F(\omega)$ and $E(\omega)$ in ω.

If c^4 is insensible, but c^2 just sensible,

$$(1 - c^2\sin^2\omega)^{\mp\frac{1}{2}} = 1 \pm \tfrac{1}{2}c^2\sin^2\omega = 1 \pm \frac{c^2}{4}(1 - \cos 2\omega).$$

Hence
$$F(\omega) = \left(1 + \frac{c^2}{4}\right)\omega + \frac{c^2}{8}\sin 2\omega;$$

$$E(\omega) = \left(1 - \frac{c^2}{4}\right)\omega - \frac{c^2}{8}\sin 2\omega.$$

When on the contrary c is so near to 1, that b^2 is insensible,

$$c^2 = 1 - b^2 = 1,\quad \Delta = \cos\omega,$$

then
$$F(\omega) = \int_0 \sec\omega\,.\,d\omega,$$

which Mr Cayley calls *Gudermann's* integral;

and
$$E(\omega) = \int \cos\omega\, d\omega = \sin\omega.$$

But if at the same time we make $\omega = \frac{1}{2}\pi$, F_c is logarithmic infinity. It is important to value it when b is sensible, though b^2 may be negligible. I find the simplest method thus:

Let $$M = \int_0 \frac{c \sin \omega \, d\omega}{\Delta}$$

and $$N = \int_0 \frac{1 - c \sin \omega}{\Delta} d\omega,$$

then $$M + N = F(\omega).$$

When b^2 is negligible, $c = 1$, $\Delta = \cos \omega$, and N becomes

$$\int_0 \frac{1 - \sin \omega}{\cos \omega} \, . \, d\omega = \int_0 \frac{\cos \omega}{1 + \sin \omega} \, . \, d\omega = \int_0 \frac{d \sin \omega}{1 + \sin \omega} = \log (1 + \sin \omega),$$

which mounts to log 2 when $\omega = \frac{1}{2}\pi$. Further

$$M = \int \frac{-c \, . \, d \, . \cos \omega}{\sqrt{(b^2 + c^2 \cos^2 \omega)}}$$

as in the preceding article; or

$$M = \log \frac{1 + c}{\sqrt{(b^2 + c^2 \cos^2 \omega)} + c \cos \omega},$$

and when $$\omega = \tfrac{1}{2}\pi, \quad M = \log \frac{1 + c}{b},$$

or here $$= \log \frac{2}{b}.$$

Thus total $F(\omega)$ when ω reaches $\frac{1}{2}\pi$, and b^2 is insensible, is found to be

$$M + N \text{ or } \log \frac{2}{b} + \log 2 \text{ or } \log \frac{4}{b}.$$

A cardinal result.

Cor. From $F_c = \log \frac{4}{b}$ when c^2 vanishes and b is sensibly $= 1$, we deduce that $F_b = \frac{4}{c}$, when b^2 vanishes and c is sensibly $= 1$. Or if we write by preference $\frac{1}{2}\pi \, . \, C$ for F_c, and $\frac{1}{2}\pi \, . \, B$ for F_b; then $\frac{1}{2}\pi B = \log \frac{4}{c}$ when c is evanescent.

Direct Series for $F(\omega)$ and $E(\omega)$.

8. Developing Δ^{-1} by Binomial Theorem when c^2 is not very near to 1, we have

$$d\,.\,F(c\omega)=\left\{1+\tfrac{1}{2}c^2\sin^2\omega+\frac{1\,.\,3}{2\,.\,4}c^4\sin^4\omega+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}c^6\sin^6\omega+\text{\&c.}\right\}d\omega,$$

whence $$F(c\omega)=\omega+k_1c^2\Omega_1+k_2c^4\Omega_2+k_3c^6\Omega_3+\text{\&c.}$$

if Ω_n stands for $\int_0(\sin\omega)^{2n}d\omega$, of which we ought to find some upper limit when n is large. Multiplying by $\sec\omega\,.\,\cos\omega=1$ under $\int$, you find $\Omega_n=\int_0\sec\omega\,.\,(\sin\omega)^{2n}d\sin\omega$. Integrating by the formula

$$\int udz=uz-\int zdu,$$

you get $$\Omega_n=\sec\omega\,.\,\frac{(\sin\omega)^{2n+1}}{2n+1}-\int\frac{(\sin\omega)^{2n+1}}{2n+1}d\sec\omega,$$

and when ω is less than $\frac{1}{2}\pi$, the last integral is *positive*. Also

$$\sec\omega\,.\,\sin\omega=\tan\omega.$$

Thus Ω_n is always less than

$$\tan\omega\,.\,\frac{(\sin\omega)^{2n}}{2n+1}.$$

Yet our series is too laborious for use, unless special values of c and ω favour.

The series for $E(c,\omega)$ follows easily by

$$F-E=\int\left(\frac{1}{\Delta}-\Delta\right)d\omega=\int(c^2\sin^2\omega)\,\Delta^{-1}\,d\omega$$

$$=\int\{c^2\sin^2\omega+k_1c^4\sin^4\omega+k_2c^6\sin^6\omega+\text{\&c.}\}\,d\omega$$

$$=c^2\,.\,\Omega_1+k_1c^4\Omega_2+k_2c^6\Omega_3+k_3c^8\Omega_4+\text{\&c.}$$

9. In the particular case of $\omega=\frac{1}{2}\pi$, these two series give values not despicable. For the equation $2n\,.\,\Omega_n=(2n-1)\,.\,\Omega_{n-1}-\cos\omega\,.(\sin\omega)^{2n-1}$ justifies itself by mere differentiation, since also every term vanishes when $\omega=0$. Put $\omega=\frac{1}{2}\pi$, $\cos\omega=0$, and the last term vanishes. Thus we obtain $$\Omega_n{}'=\frac{2n-1}{2n}\,\Omega'_{n-1},$$

if the accent on Ω denote that ω is to be pushed to the value of $\frac{1}{2}\pi$.

Now
$$\Omega_0 = \int_0 d\omega = \omega, \text{ or } \Omega_0' = \tfrac{1}{2}\pi;$$

$$\therefore\quad \Omega_1' = \tfrac{1}{2}\Omega_0' = \tfrac{1}{2}(\tfrac{1}{2}\pi);\quad \Omega_2' = \tfrac{3}{4}\cdot\tfrac{1}{2}(\tfrac{1}{2}\pi);\quad \Omega_3' = \tfrac{5}{6}\cdot\tfrac{3}{4}\cdot\tfrac{1}{2}(\tfrac{1}{2}\pi);$$

thus in general $\Omega_n' = k_n \cdot \frac{1}{2}\pi$. Finally

$$F_c = \tfrac{1}{2}\pi\,\{1 + k_1^2c^2 + k_2^2c^4 + k_3^2c^6 + \ldots k_n^2c^{2n} + \ldots\},$$

and
$$F_c - E_c = \tfrac{1}{2}\pi\,\{k_1c^2 + k_1k_2c^4 + k_2k_3c^6 + \ldots + k_{n-1}k_nc^{2n} + \ldots\}.$$

It is something to have F_c before us as an algebraic series in rising powers of c^2. It is convenient to put $F_c = \frac{1}{2}\pi \,.\, C$. I call C the *Modular.*

Thus
$$C = 1 + \left(\frac{1}{2}\right)^2 c^2 + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2 c^4 + \left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2 c^6 + \&c.\ldots$$

But all the terms being positive, no approximation is easily gained as to the value of C from a given value of c^2. Since

$$k_{n-1} - k_n = \left(\frac{2n}{2n-1} - 1\right) k_n = \left(\frac{1}{2n-1}\right) k_n;$$

we find
$$E_c = \tfrac{1}{2}\pi\,\{1 - k_1^2c^2 - \tfrac{1}{3}k_2^2c^4 - \tfrac{1}{5}k_3^2c^6 - \&c.\ldots\}.$$

Better series for F_c and E_c.

10. In $\Delta(\omega)$ we may replace the trigonometrical element by algebraic terms in two obvious ways, either from $v = \sin\omega$ or from $t = \tan\omega$, yielding

$$d\omega = \frac{d\omega}{\sqrt{(1-v^2)}},\quad d\omega = \frac{dt}{1+t^2}.$$

The former makes

$$F(c,\omega) = \int_0 \frac{dv}{\sqrt{(1-v^2)}\sqrt{(1-c^2v^2)}},\quad E(c,\omega) = \int_0 \sqrt{\frac{1-c^2v^2}{1-v^2}}\,.\,d\omega.$$

From the latter,

$$\Delta^2(c,\omega) = 1 - (1-b^2)\sin^2\omega = \cos^2\omega + b^2\sin^2\omega = \cos^2\omega\,(1+b^2t^2) = \frac{1+b^2t^2}{1+t^2},$$

whence

$$F(c,\omega) = \int_0 \frac{dt}{\sqrt{(1+t^2)}\sqrt{(1+b^2t^2)}},\text{ and } E(c,\omega) = \int_0 \sqrt{\frac{1+b^2t^2}{1+t^2}}\cdot\frac{dt}{1+t^2}.$$

Call the two last for a moment $\phi(b, t)$ and $\psi(b, t)$. Then nothing forbids our supposing b^2 to exceed 1.

We may inquire what will result by changing b and t into *their reciprocals.*

$$\phi(b^{-1}t^{-1}) \text{ must be } \frac{d\,.\,t^{-1}}{\sqrt{(1+t^{-2})}\sqrt{(1+b^{-2}t^{-2})}} \text{ or } \frac{-dt}{\sqrt{(t^2+1)}\sqrt{(t^2+b^{-2})}}$$
$$=\frac{-bdt}{\sqrt{(1+t^2)}\sqrt{(b^2t^2+1)}},$$

which reproduces

$$-b\,.\,d\phi(b,\ t),\ \text{so that}\ d\phi(b\,.\,t)+b^{-1}\,.\,\phi(b^{-1}\,.\,t^{-1})=0,$$

whence by integration

$$\phi(b,\ t)+b^{-1}\,.\,\phi(b^{-1},\ t^{-1})=C'\ (\textit{constant}).$$

Again,

$$d\psi(b^{-1},\ t^{-1})=\sqrt{\frac{1+b^{-2}t^{-2}}{1+t^{-2}}}\cdot\frac{dt^{-1}}{1+t^{-2}}=-\sqrt{\frac{b^2t^2+1}{b^2(t^2+1)}}\cdot\frac{dt}{t^2+1}$$
$$=-\frac{1}{b}\,.\,d\psi(b,\ t),$$

which integrated yields

$$\psi(b,\ t)+b\,.\,\psi(b^{-1}t^{-1})=C''\ (\textit{second constant}).$$

Suppose c' the modulus for which b^{-1} is *sub*modulus, or

$$c'^2+b^{-2}=1,\ \therefore\ c'^2=1-b^{-2}=\frac{-c^2}{b^2}.$$

Let $\tan\theta=t^{-1}$, (or $\theta+\omega=\frac{1}{2}\pi$), then, to fix the constants of integration in

$$\left.\begin{array}{ll} & F(c,\ \omega)+b^{-1}.\,F(c',\ \theta)=C', \\ \text{and} & E(c,\ \omega)+bE(c'\,\theta)\quad\ =C'' \end{array}\right\},\qquad \begin{array}{l}\text{make}\ \ \theta=0,\quad \omega=\tfrac{1}{2}\pi; \\ \therefore\quad C'=F_c,\ C''=E_c.\end{array}$$

After this, take $\omega=0$, $\theta=\frac{1}{2}\pi$; whence

$$b^{-1}F(c'\,\theta)=F_c;\quad bE(c'\,\theta)=E_c.$$

It remains only, when we take $\theta=\frac{1}{2}\pi$, to develop by the series for F_c and E_c in Art. 9, merely replacing c^2 by c'^2, that is, by $\frac{-c^2}{b^2}$. Hence we obtain

$$F_c=\tfrac{1}{2}\pi\,.\,b^{-1}\left\{1-k_1^2\,.\,\frac{c^2}{b^2}+k_2^2\,.\,\frac{c^4}{b^4}-k_3^2\,.\,\frac{c^6}{b^6}+\&c.\right\},$$

$$E_c=\tfrac{1}{2}\pi\,.\,b\ \ \left\{1+k_1^2\,.\,\frac{c^2}{b^2}-\tfrac{1}{3}k_2^2\,.\,\frac{c^4}{b^4}+\tfrac{1}{5}k_3^2\,.\,\frac{c^6}{b^6}-\&c.\right\}.$$

In these, the terms being of alternate sign, we can get approximations, and know the limits of error without great labour. Thus one step forward has been made. *Our first task* is, to compute the *complete* integrals F_c, E_c, or their ratio to $\frac{1}{2}\pi$.

To make the last series of avail, c^2 at its maximum must not exceed b^2, i.e. must not exceed $\frac{1}{2}$. At the worst convergence, $c=\frac{1}{2}=b^2$, we easily find the limits of error in taking an odd or an even number of terms.

The student may compare the new method with the former in the case of $c=\frac{1}{2}$, $c^2=\frac{1}{4}$, $b^2=\frac{3}{4}$ from Art. 9,

$$F_c = \tfrac{1}{2}\pi\left\{1+(\tfrac{1}{2})^2 . 4^{-1} + \left(\frac{1}{2}\cdot\frac{3}{4}\right)^2 . 4^{-2} + \left(\frac{1.3.5}{2.4.6}\right)^2 . 4^{-3} + \&c.\right\},$$

but by *this* Article

$$= \tfrac{1}{2}\pi . \sqrt{\tfrac{4}{3}}\left\{1-(\tfrac{1}{2})^2 . 3^{-1} + \left(\frac{1.3}{2.4}\right)^2 . 3^{-2} - \left(\frac{1.3.5}{2.4.6}\right)^2 . 3^{-3} + \&c.\right\}.$$

With c variable.

11. By inspection of the series for F_c and $F_c - E_c$ in Art. 9, we readily discover the equations

$$\int_0 c\,.\,d\,(cF_c) = F_c - E_c = -c\,.\,\frac{dE_c}{dc},$$

so that either of the two series can be reproduced by the other. This suggests to differentiate the original definition of F and E with the highest value of ω *fixed*, but with c variable. We must differentiate under the sign $\int$.

Then $\dfrac{dE}{dc} = \displaystyle\int_0 \frac{d}{dc}.\,\Delta\,.\,d\omega = \int_0 \frac{-c\sin^2\omega}{\Delta}\,d\omega$. Multiply by $-c$;

$$\therefore\quad -c\,\frac{dE}{dc} = \int_0 \frac{c^2\sin^2\omega}{\Delta}\,d\omega = F - E;\ \text{even with } \omega \textit{ indefinite.}$$

Next $\dfrac{d\,(cF)}{dc} = \dfrac{d}{dc}\displaystyle\int_0 (c^{-2}-\sin^2\omega)^{-\frac{1}{2}}\,d\omega = \int_0 \frac{d}{dc}(c^{-2}-\sin^2\omega)^{-\frac{1}{2}}\,d\omega$

$$= \int_0 \frac{c^{-3}}{(c^{-2}-\sin^2\omega)^{\frac{3}{2}}}\,d\omega = \int_0 \frac{d\omega}{\Delta^3}.$$

Consult the Table of Reduction in Article 4, and you find (only multiplying by b^2)

$$b^2 . \frac{d(cF)}{dc} = E - c^2 . \frac{\sin\omega . \cos\omega}{\Delta}, \text{ which when } \omega = \tfrac{1}{2}\pi, \text{ gives}$$

$$b^2 . \frac{d(cF_c)}{dc} = E_c \ldots\ldots\ldots\ldots\ldots\ldots(a).$$

Two of such relations between F and E ought to suffice; but their variety is enticing and bemazing, when $\omega = \frac{1}{2}\pi$.

Equation (a) by expansion of $d(cF_c)$ yields

$$E_c - b^2F_c = b^2c . \frac{dF_c}{dc}.$$

Thus $$\frac{dF_c}{dc} = \frac{E_c - b^2F_c}{b^2c}, \text{ and } \frac{-dE_c}{dc} = \frac{F_c - E_c}{c}.$$

Among other curiosities, we now for $d\aleph_c$ find

$$F_c^2 . \frac{d\aleph_c}{dc} = F_c . \frac{dE_c}{dc} - E_c . \frac{dF_c}{dc}$$

or $$F_c^2 . b^2c \frac{d\aleph_c}{dc} = -b^2F_c(F_c - E_c) - E_c(E_c - b^2F_c)$$

$$= -b^2F_c^2 + 2b^2F_cE_c - E_c^2.$$

Divide by $-F_c^2$;

$$\therefore\ -b^2c^2 . \frac{d\aleph_c}{cdc} = b^2 - 2b^2\aleph_c + \aleph_c^2 = b^2c^2 + (\aleph_c - b^2)^2.$$

That, with ω indefinite, $E - b^2F$ is positive, is not an obvious truth; though the 5th integral in the Table of Art. 4 proves it, when ω is less than $\frac{1}{2}\pi$.

It is not amiss to add that $E_c - b^2F_c = c^2F_c - (F_c - E_c)$; which, by our series in Art. 9,

$$= \tfrac{1}{2}\pi\,\{k_1c^2 + \tfrac{1}{3}k_1k_2c^4 + \tfrac{1}{5}k_2k_3c^6 + \&c.$$

12. Since $c^2 + b^2 = 1$, $cdc + bdb = 0$. Differentiate

$$\int_0 cd(cF_c) = F_c - E_c, \text{ then } c^2dF_c + F_c . cdc = dF_c - dE_c,$$

whence $dE_c = (1 - c^2)\,dF_c - cdc . F_c = b^2 . dF_c + bdb . F_c = b . d(bF_c)$,

and by integration, $$E_c = \int b . d(bF_b) \ldots\ldots\ldots\ldots\ldots\ldots(l),$$

in which we have now passed to b as the leading variable.

Again, since we also had $F_c - E_c = -c \cdot \frac{dE_c}{dc}$; take the value of F_c hence given, and insert it in the last term of $b^2 dF_c = dE_c + F_c \cdot cdc$ just now attained, then

$$b^2 \cdot dF_c = dE_2 + cdc\left(E_c - c \cdot \frac{dE_c}{dc}\right) = (1-c^2)dE_c + cdc \cdot E_c = b^2 dE_c - bdb \cdot E_c.$$

Divide by b^2, $\quad\therefore\ dF_2 = dE_2 - E_c \cdot b^{-1}\, db.$

Finally integrating, $\quad F_c = \int b \cdot d\,(b^{-1} \cdot E_c)$ ....(m).

If either E_c or F_c were known in functions of b, the equations (l), (m) would enable us to deduce the other. But when we seek for the *indefinite* integrals F, E in rising powers of b, we are cast upon $\sqrt{(1+b^2t^2)}$ as the surd instead of Δ, and t (or $\tan\omega$) increases towards infinity when ω approaches $\frac{1}{2}\pi$; a fact which stops us. Nor does the artifice of Art. 7 (in which $F = M + N$) avail us. For though the integral M is found in finite known terms, yet

$$N = \int_0 \frac{1 - c\sin\omega}{\Delta}\, d\omega = \int_0 \sqrt{\frac{1 - c\sin\omega}{1 + c\sin\omega}}\, d\omega.$$

We easily found $N = \log(1 + \sin\omega)$ when $c = 1$, but when b is not so small that b^2 is negligible, to find N even at the extreme value $\omega = \frac{1}{2}\pi$ is not obvious.

Case of F_c and E_c when b^2 is less than c^2.

13. We may begin from $b = \text{zero}$, $E_c = 1$, and work upwards with b increasing, by the equations (l), (m) of last Article. When b^2 is omissible, $F_c = \log\frac{4}{b}$. These are our starting points.

Insert the latter in our equation (l), then

$$E_c = \int_0 b \cdot d\left(b \log\frac{4}{b}\right) = 1 + \frac{b^2}{2}\left(\log\frac{4}{b} - \frac{1}{2}\right),$$

in which E_c is found to take cognizance of b^2, while before we had only $E_c = 1$. Integrating by equation (l) we raise the order of accuracy from b^n to b^{n+2}, as the two factors b and b under $\int$ denote.

When we proceed to obtain F_c by equation (m) the factors b and $-b$ neutralize each other in respect to the *order* to which b rises. Inserting in (m) the values of E_c just obtained, we have

$$F_c=\int b\,.\,d\left[b^{-1}.\left\{1+\frac{b^2}{2}\left(\log\frac{4}{b}-\frac{1}{2}\right)\right\}\right]=\int b\,.\,d\left\{b^{-1}+\frac{b}{2}\left(\log\frac{4}{b}-\frac{1}{2}\right)\right\}.$$

This is integrable, and gives

$$F_c=\log\frac{4}{b}+\frac{b^2}{4}\left(\log\frac{4}{b}-1\right),\ \text{if } b^4,\ b^6 \text{ are omissible.}$$

Next repeating the use of (l) and giving in it to F_c the value last obtained, we deduce a new value of E_c in which b^4 appears, and only b^6, b^8 are supposed too small to retain.

The law of the terms begins to appear: namely b^{2n} has a coefficient of the form

$$M_n\log\frac{4}{b}\pm P_n,$$

in which we must try to fix the value of M_n and P_n.

14. We can now simplify our problem. Let F_c, true to the order b^{2r}, be called f_r, and E_c similarly be e_r, and write

$$f_r=\log\frac{4}{b}+\beta_1+\beta_2+\ldots+\beta_r;\quad e_r=1+\alpha_1+\alpha_2+\ldots+\alpha_r;$$

where the subscript indices denote the *order* in b^2; that is, in β_n and α_n the order will be b^{2n}. When new final terms β_{r+1}, α_{r+1} are added the totals are f_{r+1}, e_{r+1}.

Evidently then the equations (l), (m) will hold good, if for E_c, F_c we substitute any two final terms α_r, β_r, in order to deduce β_{r+1}, α_{r+1}. If full proof is desired, write

First $$e_r=\int b\,.\,d\,(b\,.f_{r-1})\ \text{and}\ f_r=\int b\,.\,d\,(b^{-1}\,.\,e_r).$$

Next $$e_{r+1}=\int b\,.\,d\,(b\,.f_r)\ \text{and}\ f_{r+1}=\int b\,.\,d\,(b^{-1}\,.\,e_{r+1}),$$

then since $$e_{r+1}-e_r=\alpha_{r+1},\ \text{and}\ f_{r+1}-f_r-\beta_{r+1},$$
we replace (l) and (m) by

$$\alpha_{r+1}=\int b\,.\,d\,.\,(\beta_r)\ \text{and}\ \beta_{r+1}=\int bd\,(b^{-1}\,.\,\alpha_{r+1}).$$

Suppose we have found

$$\beta_{r-1}=M_{r-1}\left(\log\frac{4}{b}-m_{r-1}\right)b^{2n-2}$$

as known value, from it we deduce

$$\alpha_r = \int b \,.\, d\,(b\beta_{r-1}) = P_r \left(\log \frac{4}{b} - n_r\right) b^{2n},$$

by integration when β_{r-1} receives its value. This requires

$$2r \,.\, P_r = (2r-1)\, M_{r-1}$$

and
$$n_r + \frac{1}{2r} = m_{r-1} + \frac{1}{2r-1},$$

or
$$P_r = \frac{2n-1}{2r} \,.\, M_{r-1}$$

and
$$n_r = m_{r-1} + \frac{1}{(2r-1)\,2r}.$$

Repeat the process to find β_r from α_r and you get

$$M_r = \frac{2r-1}{2r} P_r;$$

$$m_r = n_r + \frac{1}{(2r-1)\,2r}.$$

Thus the law of continuation is known to an indefinite extent; and we easily deduce

$$M_r = k_r^2, \quad P_r = k_{r-1} k_r,$$

$$m_r = \frac{2}{1\,.\,2} + \frac{2}{3\,.\,4} + \ldots + \frac{2}{(2r-1)\,2r}$$

which converges to log 2 and

$$n_r = m_r - \frac{1}{(2r-1)\,2r}.$$

In brief

$$\left.\begin{aligned} F_c &= \log \frac{4}{b} + \Sigma\, k_r^2 \,.\left(\log \frac{4}{b} - m_r\right) b^{2r} \\ E_c &= 1 + \Sigma k_{r-1} k_r \,.\left(\log \frac{4}{b} - n_r\right) b^{2r} \end{aligned}\right\} \text{ in which } r \text{ represents } 1, 2, 3, 4\ldots \text{ indefinitely.}$$

Evidently this converges when b^2 is as great as $\frac{1}{2}$. It is some satisfaction even *in theory* thus to perfect what was lacking in Art. 10. If these are pushed to $\frac{1}{2}$, making thereby $b^2 = \frac{1}{2}$, the whole interval from $c = 0$ to $c = 1$, would be filled. But higher and better methods quickly supersede this process, which rather belongs to the infancy of our topic.

15. The two equations which open Art. 11 may be presented very differently, as equations (l) and (m) in Art. 12 show. Of course only two can be independent. To *remember* all the forms would seldom repay a student: yet it may be worth while to add a few more.

$$E_c - b^2F_c = b^2c \cdot \frac{dF_c}{dc} \qquad\qquad (1)$$

from (a) in Art. 11.

Again by Art. 11 $\quad F_c - E_c = \int\left(cd\,(cF_c)\right)$

$$= \int_0 c^2dF_c + \int_0 F_c cdc.$$

Subtract this last from the identical equation

$$c^2F_2 = \int_0 c^2dF_c + \int F_c \cdot 2cdc,$$

then

$$E_c - b^2F_c = \int_0 F_c \cdot cdc \qquad\qquad (2).$$

Between (1) and (2) eliminate the left member,

then

$$b^2c^2 \cdot \frac{dF_c}{cdc} = \int F_c \cdot cdc \qquad\qquad (3),$$

in which neither E_c nor dE_c remains.

Moreover, it yields, since $cdc = -\,bdb$,

$$b^2c^2 \cdot \frac{dF_c}{bdb} = \int_0 F_c \cdot bdb \qquad\qquad (4);$$

and, exchanging c with b,

$$c^2b^2 \cdot \frac{dF_b}{cdc} = \int_0 F_b \cdot cdc \qquad\qquad (5),$$

which shows, compared to (3), that F_c and F_b fulfil the *same* differential equation of c in the second order.

16. In general, if b be small, and $\tan\omega = t$,

$$\Delta = \sqrt{(1 - \overline{1 - b^2} \cdot \sin^2\omega)} = \sqrt{(\cos^2\omega + b^2\sin^2\omega)} = \cos\omega \cdot \sqrt{(1 + b^2t^2)},$$

and the surd may be developed in powers of b^2t^2 by Binomial Theorem.

Also we have

$$E = \int_0 \Delta d\omega = \int_0 \cos\omega \cdot \sqrt{(1 + b^2t^2)}\, d\omega = \int\sqrt{(1 + b^2t^2)} \cdot d\sin\omega,$$

$$(F - E) = \int_0 (c^2\sin^2\omega \cdot \Delta^{-1}\, d\omega) = c^2 \cdot \int_0 \sin^2 \cdot \frac{(1 + b^2t^2)^{-\frac{1}{2}}}{\cos\omega} \cdot d\omega$$

$$c^2 = \int_0 t^2 \cdot (1 + b^2t^2)^{-\frac{1}{2}} \cdot d\sin\omega.$$

If each surd be expanded, and T_n represent $\int_0 (\tan\omega)^{2n}\, d\sin\omega$, we by mere routine find

$$E = \sin\omega + k_1 \,.\, b^2 T_1 - \tfrac{1}{3}k_2 \,.\, b^4 T_2 + \tfrac{1}{5}k_3 \,.\, b^6 T_3 - \&c.,$$

$$F - E = c^2 \,.\, \{T_1 - k_1 b^2 T_2 + k_2 b^4 T_3 - k_3 b^6 T_4 + \&c.\}.$$

But to obtain good convergence you need bt small at the upper limit, though $t\,(=\tan\omega)$ becomes indefinitely large when ω approaches $\frac{1}{2}\pi$. Moreover the integral T_n is clumsy enough. These two series can seldom be of avail for computing; but they show us the *tendency* of E and of $F - E$, when ω is $< 45°$, $\therefore t < 1$, and when b is very small.

Mere differentiation justifies the equation

$$(2n+1)\, T_n + 2n \,.\, T_{n+1} = (\tan\omega)^{2n} \,.\, \sin\omega,$$

by which the series of T may be continued upward or downward.

When ω is $< 45°$, the series of T diminishes, and from

$$T_n = \frac{\sin\omega t^{2n}}{2n+1} - \frac{2n}{2n+1} \,.\, T_{n+1}$$

by constantly adding 1 to n, you can deduce

$$T_n = \frac{\sin\omega}{2n+1} \,.\, t^{2n} \,.\, \left\{1 - \frac{2n}{2n+3} \,.\, t^2 \,.\, \left\{1 - \frac{2n+2}{2n+5} \,.\, t^2 \,.\, \left\{1 - \&c.\ \&c.\right.\right.\right.$$

Suppose b^4 and b^4t^4 omissible; also

$$T_1 = \int_0 t^2 \cos\omega\, d\omega = \int_0 (\sec^2\omega - 1)\cos\omega\, d\omega = \int_0 (\sec\omega - \cos\omega)\, d\omega$$

$$= \int_0 \sec\omega\, d\omega - \sin\omega.$$

The former integral is a leader in Anticyclics.

For conciseness let S (for a moment) stand for

$$\int_0 \sec\omega\, d\omega, \qquad \therefore\ T_1 = S - \sin\omega.$$

Then $E = \sin\omega + \frac{1}{2}b^2 T_1 = \sin\omega + \frac{1}{2}b^2(S - \sin\omega) = (1 - \frac{1}{2}b^2)\sin\omega + \frac{1}{2}b^2 S.$

Also $$F - E = (1 - b^2)(T_1 - \tfrac{1}{2}b^2 T_2),$$

whence $$F = (1 - \tfrac{1}{2}b^2)\sin\omega + \tfrac{1}{2}b^2 S + (1 - b^2)(S - \sin\omega) - \tfrac{1}{2}c^2 b^2 T_2$$
$$= \tfrac{1}{2}b^2 \sin\omega + (1 - \tfrac{1}{2}b^2)\, S - \tfrac{1}{2}c^2 b^2 T_2.$$

If we need only to find b^2F, *when b^4 is omissible*, we may be satisfied with $b^2F = b^2S$.

CHAPTER II.

METHOD OF MULTIPLE ARCS.

1. IN Astronomy the integrals of the family connected with the surd Δ are treated by multiple arcs, which makes it reasonable to give it a place here, where indeed it is both simple, and fertile of results. Legendre thought it worthy of immense labour, and his work on it is a κτῆμα ἐς ἀεί.

Since $2\sin^2\omega = 1 - \cos 2\omega$, we have a right to assume with unknown constants μ and h, Δ^2 or $1 - c^2 \sin^2 \omega$ identically

$$= \mu^2 (1 + 2h\cos 2\omega + h^2).$$

To determine the new constants, first make $\omega = 0$; then

$$\mu = (1 + h)^{-1}.$$

Next, make $\omega = \frac{1}{2}\pi$, then

$$1 - c^2 = \mu^2 (1 - h)^2 \text{ or } b = \mu (1 - h).$$

Hence $b = \dfrac{1-h}{1+h}$, and conversely

$$h = \frac{1-b}{1+b} \text{ or } (1+h)(1+b) = 2.$$

Also $$\mu = (1+h)^{-1} = \tfrac{1}{2}(1+b).$$

Again, $$c = \sqrt{1-b^2} = \frac{2\sqrt{h}}{1+h}.$$

2. If further we assume a new constant g, making $h^2 + g^2 = 1$, since h is less than 1, then symmetrically

$$h = \left(\frac{c}{1+b}\right)^2, \quad b = \left(\frac{g}{1+h}\right)^2.$$

Thus b is related to h and g, exactly as is h to b and c. Since $(1-b)^2$ is positive, $(1+b)^2$ or $(1-b)^2 + 4b$ *is greater than* $4b$.

The equation $h = \left(\frac{c}{1+b}\right)^2$ shows that h is *less* than $\frac{c^2}{4b}$. Also $(1+b)^2$ is less than 4; therefore h is *greater* than $\frac{c^2}{4}$.

Next, put $v + v^{-1} = 2\cos 2\omega$, $\therefore\ \Delta^2 = \mu^2 (1 + h\,.\,\overline{v+v^{-1}}\,. + h^2)$,

and $$\mu^{-2}\,.\,\Delta^2 = (1+hv)(1+hv^{-1});$$

whence we must seek Δ^{-1} in series

$$\mu\Delta^{-1} = (1+hv)^{-\frac{1}{2}}\,.\,(1+hv^{-1})^{-\frac{1}{2}},$$

and each surd must be expanded by Binomial Theorem.

In general $(1+u)^{-\frac{1}{2}} = 1 - k_1 u + k_2 u^2 - k_3 u^3 + \&c.$

For u assume first hv, next hv^{-1}, and multiply the results together.

The product, on mere algebraic inspection, shows the form

$$P_0 - P_1 (v + v^{-1}) + P_2 (v^2 + v^{-2}) - P_3 (v^3 + v^{-3}) + \&c.$$

where
$$\left|\begin{array}{l} P_0 = 1 + k_2{}^2 h^2 + k_2{}^2 h^4 + k_3{}^2 h^6 + \&c. \\ P_1 = k_1 h + k_2 k_1 h^3 + k_3 k_2 h^5 + k_4 k_3 h^7 + \&c. \\ P_2 = k_2 h^2 + k_3 k_1 h^4 + k_4 k_2 h^6 + k_5 k_3 h^8 + \&c. \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ P_n = k_n h^n + k_{n+1} h_1\,.\,k^{n+2} + k_{n+2} k_2\,.\,h^{n+4} + \&c. \end{array}\right|$$

If then we account every P to be known, restore to $v^n + v^{-n}$ its value $2\cos 2nv$, you obtain

$$\mu\Delta^{-1} = P_0 - 2P_1 \cos 2\omega + 2P_2 \cos 4\omega - 2P_3 \cos 6\omega + \&c.$$

Multiply by $\mu^{-1}\,d\omega = (1+h)\,d\omega$, and integrate, then

$$F(c\omega) = (1+h)\,\{P_0\omega - P_1 \sin 2\omega + \tfrac{1}{2}P_2 \sin 4\omega - \tfrac{1}{3}P_3 \sin 6\omega + \&c.\},$$

in which P_n having h^n as a factor, and h being less than $\frac{c^2}{4b}$, the convergence is good, unless c is very near to 1.

3. To calculate the functions P for *all* values of $h < 1$, must be a vast task. But when we assume $\omega = \frac{1}{2}\pi$, all of them vanish but P_0, and we obtain simply $F_c = \frac{1}{2}\pi\,.\,(1+h)\,.\,P_0$, or giving to P_0 its value in series

$$F_c = (1+h)\,.\,\tfrac{1}{2}\pi\,\{1 + k_1{}^2 h^2 + k_2{}^2 h^4 + k_3{}^2 h^6 + \&c. \ldots\};$$

but on comparing the series with that of F_c, Ch. I., Art. 9, we find it simply means $F_c = (1+h)\,F_h$; a splendid and unsought discovery,

elicited by the mere "march-step" of the Calculus. Thus instead of calculating F_c directly, we find it *indirectly* through F_h in which h is less than c.

In Ch. I., Art. 10, we improved on the series for F_c by obtaining a new series with terms of alternate signs, but this required that c should not exceed b; so that, when $c^2 = \frac{1}{2} = b^2$, we had

$$F_c = \tfrac{1}{2}\sqrt{2}\,\{1 - k_1^{\,4} + k_2^{\,2} - k_3^{\,2} + \&c.\}.$$

But now we find

$$h = \frac{1-b}{1+b} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = \frac{(\sqrt{2}-1)^2}{2-1} = 3 - 2\sqrt{2} = \cdot 1716,$$

indeed $h^2 = (\sqrt{2}-1)^4$ which is less than $\cdot 03$. So that our first series for F_c now avails us practically.

Moreover as we deduced h from c by the law $h = \dfrac{1-b}{1+b}$, so if $h^2 + g^2 = 1$, we can get from h a smaller modulus h' by the law $h' = \dfrac{1-g}{1+g}$. To organize a *series* of moduli by the same law, it is convenient to write $c_1 b_1$ for our hg; then $c_2 b_2$ follow $c_1 b_1$ by the same law as $c_1 b_1$ follow cb; and so on to $c_3 b_3 \dots$ and $c_n b_n$.

Landen's Scale of the Moduli.

4. If we suppose the successive moduli to have been calculated (a work which Legendre completed and abides to our hand), we lessen complication by using $c_1 c_2 c_3 \dots$ in capitals just as $F_c = \frac{1}{2}\pi C$, then $F_c = (1+h)\,F_h$, divided by $\frac{1}{2}\pi$, leaves $C = (1 + c_1)\,C_1$.

Repeated, this gives $C_1 = (1 + c_2)\,C_2$, $C_2 = (1 + c_3)\,C_3$, and so on until

$$C = (1 + c_1)(1 + c_2)(1 + c_3) \dots (1 + c_n)\,C_n.$$

Now $c\,c_1 c_2 c_3 \dots$ decrease faster than $c\,c^2 c^4 c^8 \dots$; in fact, when b is as large as $\frac{1}{2}$, $(1+b)^2 > (1 + 2b)$ or > 2, so that with n quite moderate, c_n is soon insignificant, and $C_n = 1$. Finally then, with *n infinite*, we have an *exact* result

$C = (1 + c_1)(1 + c_2)(1 + c_3)$, &c., *ad infinitum*, to determine $F_c = \frac{1}{2}\pi_1 C$.

Again, we had

$$(1+c_1)=\frac{2}{1+b}=\frac{b_1}{\sqrt{b}}; \quad \therefore \quad C=\frac{b_1}{\sqrt{b}}\cdot\frac{b_2}{\sqrt{b_1}}\cdot\frac{b_3}{\sqrt{b_2}}+\&c.,$$

or $bC=\sqrt{(bb_1b_2b_3\ldots)}$ as *final result*, unless c is *very* near to 1.

COR. Moreover $b.F_c=b(1+h)F_h=(1-h)F_h$.

When c and h *vary*, from $b=\dfrac{1-h}{1+h}$, we deduce

$$d\log b = d\log.\left(\frac{1-h}{1+h}\right),$$

or

$$\frac{db}{b}=\frac{-2dh}{1-h^2}=\frac{-2dh}{g^2}=\frac{-2hdh}{hg^2}=\frac{+2gdg}{hg^2}=\frac{2dg}{hg};$$

$$\therefore \quad \frac{dg}{g}=\frac{h}{2}\cdot\frac{db}{b}.$$

Addendum.

Legendre has calculated Landen's scale

$$cc_1c_2c_3c_4\ldots, \quad bb_1b_2b_3b_4\ldots.$$

He *never* needs in practice to go farther than c_4. His peculiar treatment deserves notice.

The primary relation

$$h=\frac{1-b}{1+b} \text{ or } =\frac{(1-b)^2}{1-b^2},$$

gives

$$h=\frac{1-2b+b^2}{c^2}=\frac{(2-c^2)-2\sqrt{(1-c^2)}}{c^2}.$$

Develop the surd by Binomial Theorem; therefore

$$h=\frac{1}{4}c^2+\frac{1.3}{4.6}c^4+\frac{1.3.5}{4.6.8}c^8+\&c.$$

But easily to find limits, Legendre demands series with *alternate signs*. He ingeniously assumes a sort of *inter*modulus

$$m=\frac{c^2}{2b},$$

observing that

$$h=\frac{1-b^2}{(1+b)^2}=\frac{c^2}{(1+b)^2}$$

makes h less than $\frac{c^2}{4b}$ or than $\frac{1}{2}m$. (Write now $c_1 b_1$ for h and g, also m_1 for $\frac{c_1^2}{2b_1}$.) Then since c_1^2 is less than c^2, and $2b_1$ slightly exceeds $2b$, evidently m_1 is less than m. But indeed since

$$c=\frac{2\sqrt{c_1}}{1+c_1}, \text{ and } b_1=\frac{2\sqrt{b}}{1+b},$$

$$\therefore \quad cb_1=\frac{4\sqrt{c_1 b}}{(1+c_1)(1+b)},$$

of which the denominator $=2$; therefore

$$c^2 b_1^2 = 4c_1 b, \text{ and } \frac{c^2}{2b} \textit{ or } m=\frac{2c_1}{b_1^2}:$$

whence $\qquad m_1 : m = \frac{c_1^2}{2b_1} : \frac{2c_1}{b_1^2} = b_1 c_1 : 4, \text{ or } \frac{m_1}{m}=\frac{b_1 c_1}{4}.$

We seek for series to express c_1 and b_1 *in powers of* m.

First, $\qquad m=\frac{c^2}{2b}=\frac{1-b^2}{2b}.$

Solve for b; then $\qquad b=\sqrt{(1+m^2)}-m.$

Hence too

$$b^{-1}=\sqrt{(1+m^2)}+m \quad \text{and} \quad b^{-1}+b=2\sqrt{(1+m^2)}.$$

Also $\quad c_1=\frac{c^2}{(1+b)^2}=\frac{2bm}{1+2b+b}=\frac{2m}{b^{-1}+2+b}=\frac{2m}{2\sqrt{(1+m^2)}+2},$

or again $\qquad c_1=\frac{m}{\sqrt{(1+m^2)}+1}=\frac{\sqrt{(1+m^2)}-1}{m}.$

Develop by Binomial Theorem, therefore

$$c_1=k_1 m-\tfrac{1}{3}k_2 m^3+\tfrac{1}{5}k_3 m^5-\&c. \quad \text{.................(A)},$$

with signs alternate.

Again,

$$m=\frac{2c_1}{b_1^2}, \quad \therefore \quad b_1^2=\frac{2c_1}{m} 2\{k_1-\tfrac{1}{3}k_2 m^2+\tfrac{1}{5}k_3 m^4-\&c.\}\text{.....(B)}.$$

Further, since $c_1 m=\sqrt{(1+m^2)}-1$, take *log.* and differentiate

$$d\log(c_1 m)=\frac{d.\sqrt{(1+m^2)}}{\sqrt{(1+m^2)}-1}=\frac{\sqrt{(1+m^2)}+1}{m^2}\cdot\frac{mdm}{\sqrt{(1+m^2)}}$$

$$=\{1+(1+m^2)^{-\frac{1}{2}}\}.m^{-1}dm=\{2-k_1 m^2+k_2 m^4-k_3 m^6+\&c.\}m^{-1}dm.$$

Integrate back, adding $-\log a$, as constant of integration, therefore

$$\log(c_1 m) = (2\log m - \log a) - \tfrac{1}{2}k_1 m^2 + \tfrac{1}{4}k_2 m^4 - \tfrac{1}{6}k_3 m^6 + \&c.,$$

or
$$\log\frac{c_1 a}{m} = -\tfrac{1}{2}k_1 m^2 + \tfrac{1}{4}k_2 m^4 - \tfrac{1}{6}k_3 m^6 + \&c. \quad\ldots\ldots\ldots\ldots(C).$$

But
$$m = \frac{2c_1}{b_1^2}; \quad \frac{c_1}{m} = \frac{b_1^2}{2},$$

so that the last series $= \log(\tfrac{1}{2}\,.\,b_1^2 a)$.

Make $c = 0, \quad m = 0, \quad b = 1;$

therefore $a = 2$, constant of integration; and the series (C) $= \log.b_1^2$, and we find
$$\log c_1 = \log(b_1^2) + \log(\tfrac{1}{2}m).$$

The series (A), (B), (C) have all signs alternate, which seems to have been with Legendre the main problem here.

Some other relations deserve passing notice in this scale.

Since
$$c_1 = \frac{2\sqrt{c_2}}{1+c_2}, \quad \therefore \frac{1-c_1}{1+c_1} = \left(\frac{1-\sqrt{c_2}}{1+\sqrt{c_2}}\right)^2,$$

or
$$\sqrt{b} = \frac{1-\sqrt{c_2}}{1+\sqrt{c_2}}.$$

Also
$$1 + c_1 = \frac{(1+\sqrt{c_2})^2}{1+c_2},$$

or
$$(1+c_1)(1+c_2) = (1+\sqrt{c_2})^2,$$

and since b_1 is related to b as c is to c_1, therefore
$$(1+b_1)(1+b) = (1+\sqrt{b})^2.$$

Combine $(1+c_1)(1+b) = 2$ with $(1+c_2)(1+b_1) = 2$. Take the product; therefore
$$(1+c_2)(1+c_1)(1+b_1)(1+b) = 2\,.\,2,$$

or $(1+\sqrt{c_2})^2\,.\,(1+\sqrt{b})^2 = 2\,.\,2$, or $(1+\sqrt{c_2})(1+\sqrt{b}) = 2$.

Case of F_c when c is too near to 1.

5. We had $C = (1+c_1)C_1$ and symmetrically $B_1 = (1+b)B_1$, if B is related to b, as C to c. Multiply these together, remembering that $(1+c_1)(1+b) = 2$.

Then
$$CB_1 = 2C_1 B \text{ or } \frac{B}{C} = \tfrac{1}{2}\,.\,\frac{B_1}{C_1}.$$

Repeat the last equation, then $\frac{B_1}{C_1} = \frac{1}{2} \cdot \frac{B_2}{C_2}$;

again $$\frac{B_2}{C_2} = \frac{1}{2} \cdot \frac{B_3}{C_3};$$

and so on. Then we have in succession

$$\frac{B}{C} = 2^{-1} \cdot \frac{B_1}{C_1} = 2^{-2} \cdot \frac{B_2}{C_2} = \ldots = 2^{-n} \cdot \frac{B_n}{C_n}.$$

Multiply by $\frac{1}{2}\pi$, and take a new constant ρ whose definition is

$$\rho = \tfrac{1}{2}\pi \cdot \frac{B}{C};$$

then also $$\rho = 2^{-n} \cdot \frac{B_n}{C_n} \tfrac{1}{2}\pi.$$

If we make n so large that $(c_n)^2$ is insensible, we have found $C_n =$ simply 1. In the same case (see Ch. I., Art. 7, Cor.) we have

$$\tfrac{1}{2}\pi \cdot B_n = \log \frac{4}{c_n}.$$

These yield $$\rho = 2^{-n} \cdot \log \frac{4}{c_n}.$$

More accurately then ρ is the *limit* to which the series

$$\log \frac{4}{c};\ 2^{-1} \cdot \log \frac{4}{c_1};\ 2^{-2} \cdot \log \frac{4}{c_2};\ 2^{-3} \cdot \log \frac{4}{c_3};\text{—tend.}$$

Legendre found that at worst his limit was reached when $n = 4$. Thus, by means of the *scale* (if known), ρ is calculable from c. Thus also B becomes known from the equation

$$B = \frac{C\rho}{\frac{1}{2}\pi} \text{ and } F_b = \tfrac{1}{2}\pi B = C\rho.$$

Our use of this now supposes c small, and instead of computing F_c when c approaches 1, passes to F_b when b approaches 1, thus solving the very same problem.

The Promodulus.

6. I propose to call ρ the *Promodulus*. In the Higher Theory it becomes the leading constant, of which c, b, C, B and E_c are functions. Naturally, what ρ is to c, we denote by ρ_n to c_n. Thus $\rho_1 = \frac{1}{2}\pi \ \frac{B_1}{C_1}$.

But $\frac{B_1}{C_1}=2\frac{B}{C}$; $\therefore\ \rho_1=2\rho$. So $\rho_2=2\rho_1=2^2\rho$, $\rho_3=2\rho_2=2^3\rho$, and generally $\rho_n=2^n\rho$. This is the property which makes ρ a convenient standard. In Landen's scale it *doubles* itself when the index of c adds a unit. Therefore 2 is called the *index* of this scale.

If ρ' is to b, what ρ is to c, we have $\rho'=\frac{1}{2}\pi\cdot\frac{C}{B}$; $\therefore\ \rho\cdot\rho'=(\frac{1}{2}\pi)^2$; and one or other is always $>(\frac{1}{2}\pi)$.

Legendre must have calculated an entire table of ρ from a given c, with a view to obtain F_b from $C\rho$. *Perhaps* his table of ρ may be even now recovered. Not foreseeing the Higher Theory of which Jacobi was to be the discoverer, Legendre could not know the prominence which ρ was to assume, so he gave it neither a name nor a symbol. But already we may ask, Can we work backward, and from a *given* ρ compute the c?—The following process is the best at which I have arrived.

From a given ρ to compute c.

7. The equation $c=\frac{2\sqrt{c_1}}{1+c_1}$, or $\frac{4}{c}=\sqrt{\frac{4}{c_1}}\cdot(1+c)$, gives a formula of repetition by increasing the index of c after taking logarithms on both sides. Each time, also halve the result. Then we have in succession

$$\left.\begin{aligned}
\log\frac{4}{c}-2^{-1}\log\frac{4}{c_1}&=\log(1+c_1)\\
2^{-1}\log\frac{4}{c_1}-2^{-2}\log\frac{4}{c_2}&=2^{-1}\log(1+c_2)\\
2^{-2}\log\frac{4}{c_2}-2^{-3}\log\frac{4}{c_3}&=2^{-2}\log(1+c_3)\\
\cdots\cdots\cdots\cdots\cdots&\cdots\cdots\cdots\cdots\\
2^{-n+1}\log\frac{4}{c_{n-1}}-2^{-n}\log\frac{4}{c_n}&=2^{-n+1}\log(1+c_n)
\end{aligned}\right\}$$

Add all these together, and on the left side you have barely $\log\frac{4}{c}-2^{-n}.\log\frac{4}{c_n}$.

Hence, when n becomes infinite the total sum is

$$\log\frac{4}{c}-\rho=\log(1+c_1)+2^{-1}.\log(1+c_2)+2^{-2}\log(1+c_3)+\&c.\ldots$$

But $$1+c_1=\frac{C}{C_1};$$

and in general $$1 + c_n = \frac{C_{n-1}}{C_n};$$
so that the series
$$= (lC - lC_1) + 2^{-1}(lC_1 - lC_2) + 2^{-2}(lC_2 - lC_3) + \&c....,$$
where each term may simplify the preceding, whence also
$$\log\frac{4}{c} = \rho + \log C - \tfrac{1}{2}\log C_1 - \tfrac{1}{4}\log C_2 - \tfrac{1}{8}\log C_3 - \&c\ ;$$
where C_n is the same function of $2^n\rho$, as C of ρ.

Now a table of $\log C$, with ρ as argument, *from the Higher Theory*, will be easily constructed,

$\left(\text{since } \textit{there } \sqrt{C} = \frac{\operatorname{Cot}\rho}{\operatorname{Cot}2\rho}\cdot\frac{\operatorname{Cot}3\rho}{\operatorname{Cot}4\rho}\cdot\frac{\operatorname{Cot}5\rho}{\operatorname{Cot}6\rho}\ldots, \text{ in } \textit{Anticyclic} \text{ notation}\right).$

Then the above gives $\log\frac{4}{c}$ from a given ρ very rapidly. Conversely with c given, $\rho = \log\frac{4}{c} + \log\frac{\sqrt{b}}{b_1} + 2^{-1}\log\frac{\sqrt{b_1}}{b_2} + 2^{-2}$, &c.

To differentiate ρ.

8. The following process succeeds, by a sort of good luck.

When c is infinitesimal, $\rho = \log\frac{4}{c}$;

hence with n large, $\rho_n = \log\frac{4}{c_n}$ is approximate.

Now $$\rho = 2^{-n}\rho_n,$$
$$\therefore\ d\rho = 2^{-n}.d\rho_n = -2^{-n}.\frac{dc_n}{c_n},$$
whence $$-\frac{d\rho}{dc} = \frac{dc_n}{2^n.c_n}\frac{1}{dc} = \left(\frac{dc_n}{2^n dc}\right)\frac{1}{c_n}.$$

Here the fraction in parentheses has for equivalent
$$\left(\frac{dc_1}{2dc}\cdot\frac{dc_2}{2dc_1}\cdot\frac{dc_3}{2dc_2}\cdots\frac{dc_n}{2dc_{n-1}}\right).$$

Further, from $c = \frac{2\sqrt{c_1}}{1+c_1}$, differentiated logarithmically, you find
$$\frac{dc}{c} = \frac{1-c_1}{1+c_1}\cdot\frac{dc_1}{2c_1} = \frac{bdc_1}{2c_1},$$

so that $\dfrac{dc_1}{2dc} = \dfrac{c_1}{b_c}$. Introduce this equivalence into the last complex fraction, $\therefore\ -\dfrac{d\rho}{dc} = \left(\dfrac{c_1}{bc} \cdot \dfrac{c_2}{b_1c_1} \cdot \dfrac{c_3}{b_2c_2} \cdots \dfrac{c_n}{b_{n-1}c_{n-1}}\right)\dfrac{1}{c_n} = (c\,.\,b\,b_1b_2\ldots b_{n-1})^{-1}$. This is only approximate, while n is large, but finite. It becomes accurate when n is infinite.

Now by Art. 18 we know that $(bb_1b_2b_3\ldots) = b^2C^2$. Divide each side by c,
$$\therefore\ -\frac{d\rho}{cdc} = \frac{1}{c^2}\,.\,(b^2C^2)^{-1},$$
or
$$\frac{d\rho}{bdb} = \frac{1}{b^2c^2C^2},$$
which is sometimes valuable.

9. In fact, try it upon $\rho = \frac{1}{2}\pi\,.\,\dfrac{B}{C}$. Differentiate;
$$\therefore\ d\rho = \tfrac{1}{2}\pi\,.\,\frac{BdC - CdB}{C^2};$$
also
$$\tfrac{1}{2}\pi\,.\,C = F_c;$$
$$\tfrac{1}{2}\pi\,.\,B = F_b;$$
while from Art. 8
$$C^2 d\rho = \frac{bdb}{b^2c^2};$$
whence
$$\frac{bdb}{b^2c^2} \text{ or } \frac{-cdc}{b^2c^2} = BdF_c - CdF_b,$$
or
$$\frac{1}{b^2c^2} = B\frac{dF_c}{cdc} + C\frac{dF_b}{bdb} \quad \text{(a).}$$

LEMMA. *I say,* $\dfrac{dF_c}{cdc} = \dfrac{E_c - b^2F_c}{b^2c^2}$; [indeed Ch. I., Art. 11.] which by symmetry remains when b and c are exchanged.

Proof. By equation (a) in Ch. I., Art. 11, we had
$$E_c = b^2\,.\,\frac{d(cF_c)}{dc},$$
which, expanded, becomes
$$b^2\,.\,\frac{cdF_c + F_cdc}{dc} \text{ or } = b^2c\frac{dF_c}{dc} + b^2F_c;$$
whence
$$E_c - b^2F_c = b^2c^2\,.\,\frac{dF_c}{cdc};$$
as above asserted. Multiply (a) by b^2c^2; we now change equation (a) into
$$1 = B(E_c - b^2F_c) + C(E_b - c^2F_b).$$

Multiply by $\frac{1}{2}\pi$ to make uniformity;

$$\therefore\ \tfrac{1}{2}\pi = F_b(E_c - b^2F_c) + F_c(E_b - c^2F_b),$$

$$\text{or } \tfrac{1}{2}\pi = F_bE_c + E_bF_c - F_bF_c.$$

Thus we obtain Legendre's beautiful equation by mere *differentiations.*

Dividing by F_bF_c and introducing the Ancillae

$$\aleph_c = \frac{E_c}{F_c}, \quad \aleph_b = \frac{E_b}{F_b},$$

we have
$$\aleph_c + \aleph_b = 1 + \frac{\frac{1}{2}\pi}{F_c \cdot F_c} \quad \text{(b)},$$

which may be called, the equation of *Complementary Ancillae.*

---

10. Hitherto it has been shown how, when c is *given*, and we may assume that from c we may account $c_1\, b_1\, c_2\, b_2\, c_3\, b_3 \dots c_n\, b_n$ all to be known, we can deduce F_c (or rather C) in Art. 4; next, in order to pass to F_b, when $c^2 > \frac{1}{2}$, we attain the promodulus ρ from $2^{n-1}.\log\dfrac{4}{c_n}$ in Art. 5; whence further $F_b = C\rho$. But we next ask after E_c and E_b.

Reduction of E to E_h.

Now that we have $F_c = (1+h)\,F_h$, our most obvious method is, to differentiate this equation. But it must first be prepared conveniently.

Multiply it by
$$b = \frac{1-h}{1+h};$$

$$\therefore\ bF_c = (1-h)\,F_h = \frac{1-h}{g}.(gF_h) = \sqrt{\frac{1-h}{1+h}}.(gF_h) = \sqrt{b}\,(gF_h):$$

$$\therefore\ \log(bF_c) = \tfrac{1}{2}\log b + \log(gF_h) \quad \text{(m)}.$$

Now in Art. 9 we had

$$E_c - b^2F_c = b^2c^2.\frac{dF_c}{cdc} = -\,b^2c^2.\frac{dF}{bdb}.$$

Subtract each from c^2F_c;

$$\therefore\ F_c - E_c = c^2F_c + c^2b\frac{dF_c}{db} = c^2.\frac{d(bF_c)}{db}.$$

Divide by F'_c,

then $1-\aleph_c = c^2 \cdot \frac{d(bF_c)}{F_c \cdot db}$; | So too $1-\aleph_h = h^2 \cdot \frac{d(gF_h)}{F_h \cdot dg}$,

or $\frac{1-\aleph_c}{c^2} \cdot \frac{db}{b} = d\log(bF_c)$. | $\frac{1-\aleph_h}{h^2} \cdot \frac{dg}{g} = d\log(gF_h)$.

But in Cor. to Art. 4 we had

$$\frac{dg}{g} = \tfrac{1}{2}h \cdot \frac{db}{b}.$$

Hence by differentiating (m), page 27, we have

$$d\log(bF_c) = \tfrac{1}{2}\frac{db}{b} + d\log(gF_h),$$

which now becomes

$$\frac{1-\aleph_c}{c^2} \cdot \frac{db}{b} = \tfrac{1}{2}\frac{db}{b} + \frac{1-\aleph_h}{h^2}\left(\frac{h}{2} \cdot \frac{db}{b}\right).$$

Expunge the factor $\frac{db}{b}$, and multiply by c,

$$\therefore \left(\frac{1-\aleph_c}{c}\right) = \frac{c}{2} + \frac{c}{2}\left(\frac{1-\aleph_h}{h}\right) \quad \text{(n)}.$$

11. When c is small, we have from Ch. I., Art. 9,

$$E_c = \tfrac{1}{2}\pi(1-\tfrac{1}{4}c^2), \quad F_c = \tfrac{1}{2}\pi(1+\tfrac{1}{4}c^2);$$

$$\therefore \aleph_c = \frac{E_c}{F_c} = \frac{1-\tfrac{1}{4}c^2}{1+\tfrac{1}{4}c^2} = 1 - \tfrac{1}{2}c^2,$$

whence $$\frac{1-\aleph_c}{c} = \tfrac{1}{2}c,$$

which is evanescent with c. If then we continue the formula (n), resuming the notation of c_1 for h; and for a moment write R_c for $c^{-1}.(1-\aleph_c)$, the equation $R_c = \tfrac{1}{2}c\{1+R_h\}$ gives by repetition an infinite series $R_c = \tfrac{1}{2}c\{1+\tfrac{1}{2}c_1\{1+\tfrac{1}{2}c_2\{1+\tfrac{1}{2}c_3\{1+\&c....$, that is,

$$\frac{1-\aleph_c}{c} = \tfrac{1}{2}c + \tfrac{1}{4}cc_1 + \tfrac{1}{8} \cdot cc_1c_2 + \tfrac{1}{16}cc_1c_2c_3 + \&c.$$

This is our second *final* series, in which if c is $< b$, the convergence is excellent, and the Ancilla is satisfactorily ascertained.

Then from $E_c = \aleph_c \cdot F_c$, the E_c is found.

But what if c is $>b$? Then by the Complementary Equation of Art. 9, we can find $\aleph_b$ from $\aleph_c$, and thus deduce E_b, which completes our immediate problem. For, knowing $F_c F_b E_c E_b$ when c^2 varies from 0 to $\frac{1}{2}$, we virtually know F_c and E_c from $c^2=0$ to $c^2=1$.

The first real success is now attained, the values of F_c and F_b. I subjoin the Table calculated by Legendre, from the Grade γ as argument, where $\sin\gamma=c$, $\cos\gamma=b$.

From Legendre.

γ = the Grade.	Let $c=\sin\gamma$ $F_c=\frac{1}{2}\pi C$.	and $b=\cos\gamma$ $F_b=\frac{1}{2}\pi B$.
0°	1·5707 9632 6795	log infinity
1	1·5709 1595 8127	5·4349 0982 9625·
2	1·5712 7495 2372	4·7427 1726 5279
3	1·5718 7361 0514	4·3386 5397 6000
4	1·5727 1243 4995	4·0527 5816 9549
5	1·5737 9213 0925	3·8317 4199 9766
6	1·5751 1360 7777	3·6518 5596 9479
7	1·5766 7798 1593	3·5004 2249 9173
8	1·5784 8657 7689	3·3698 6802 6668
9	1·5805 4093 3896	3·2553 0294 2143
10	1·5828 4280 4338	3·1533 8525 1888
11	1·5853 9416 3775	3·0617 2861 2039
12	1·5881 9721 2527	2·9785 6895 1181
13	1·5912 5438 2014	2·9025 6494 0670
14	1·5945 6834 0932	2·8326 7258 2918
15	1·5981 4200 2113	2·7680 6314 5369
16	1·6019 7853 0087	2·7080 6761 4590
17	1·6060 8134 9410	2·6521 3800 4630
18	1·6014 5415 3790	2·5998 1973 0061
19	1·6151 0091 6068	2·5507 3144 9627
20	1·6200 2580 9124	2·5045 5007 9002
21	1·6252 3366 7759	2·4609 9945 8304
22	1·6307 2910 1631	2·4198 4165 3739
23	1·6365 1740 9336	2·3808 7019 0604
24	1·6426 0414 3713	2·3439 0472 4447

γ	$c = \sin\gamma$ F_c	$b = \cos\gamma$ F_b
25°	1·6489 9522 8479	2·3087 8679 8167
26	1·6556 9692 6310	2·2753 7642 9612
27	1·6627 1595 8491	2·2435 4934 1699
28	1·6700 5942 6270	2·2131 9469 4981
29	1·6777 3488 4081	2·1842 1321 6949
30	1·6857 5035 4813	2·1565 1564 7500
31	1·6941 1435 7306	2·1300 2143 8399
32	1·7028 3523 6412	2·1046 5765 8491
33	1·7119 2469 5156	2·0803 5806 6692
34	1·7213 9083 1374	2·0570 6232 2797
35	1·7312 4517 5657	2·0347 1531 2186
36	1·7414 9923 4427	2·0132 6656 5201
37	1·7521 6523 6469	1·9926 6975 5735
38	1·7632 5618 4059	1·9728 8226 6275
39	1·7747 8590 9104	1·9538 6480 9252
40	1·7867 6913 4885	1·9355 8109 6006
41	1·7992 2154 4050	1·9179 9754 6438
42	1·8121 5985 3662	1·9010 8303 3465
43	1·8256 0189 8136	1·8848 0865 7384
44	1·8395 6672 1094	1·8691 4754 6031
45	1·8540 7467 7301	1·8540 7467 7301

Another Method for E_c or rather $\aleph_c$.

12. When c varies, the abundance of equations is so embarrassing, that many will prefer the following treatment.

$$F - E = \int_0 \frac{c^2 \sin^2 \omega}{\Delta}\, d\omega,$$

which by expanding Δ^{-1} and observing that $\sin^2\omega = \frac{1}{2}(1 - \cos 2\omega)$, and $c^2(1+h) = 2c\sqrt{h}$, yields

$$F - E = c\sqrt{h}\int_0 (1 - \cos 2\omega)(P_0 - 2P_1 \cos 2\omega + 2P_2 \cos 4\omega - \&c.)\, d\omega$$

$$= c\sqrt{h}\,.\int_0 \left\{\begin{array}{l} P_0 - 2P_1\cos 2\omega + 2P_2\cos 4\omega - 2P_3\cos 6\omega + \&c. \\ -P_0\cos 2\omega + 2P_1(\cos 2\omega)^2 - 2P_2\cos 2\omega\cos 4\omega + \&c. \end{array}\right\} d\omega.$$

Compress by the formula

$$2\cos 2\omega \,.\, \cos 2n\omega = \cos . \overline{n-2}\omega + \cos \overline{n+2}\omega,$$

then

$$F - E = c\sqrt{h}\int_0 [(P+P_1) - (P_0 + 2P_1 + P_2)\cos 2\omega$$

$$+ (P_1 + 2P_2 + P_3)\cos 4\omega - (P_2 + 2P_3 + P_4)\cos 6\omega - \&c.] \,.\, d\omega$$

$$= c\sqrt{h}\,[(P_0 + P_1)\,\omega - \tfrac{1}{2}(P_0 + 2P_1 + P_2)\sin 2\omega$$

$$+ \tfrac{1}{4}(P_1 + 2P_2 + P_3)\sin 4\omega - \tfrac{1}{6}\ \&c.],$$

from which we obtain E by eliminating F; which when $\omega = \frac{1}{2}\pi$, leaves simply $F_c - E_c = c\sqrt{h} \,.\, (P_0 + P_1)\,\frac{1}{2}\pi$.

We found by mere inspection that $\frac{1}{2}\pi \,.\, P_0 = F_h$. Let us inspect hP_1.

From Art. 2, $hP_1 = k_1h^2 + k_2k_1h^4 + k_3k_2h^6 + k_4k_3h^8 + \&c.$, and immediately from Ch. I., Art. 9, we discover that

$$\tfrac{1}{2}\pi \,.\, hP_1 = F_h - E_h.$$

We can now eliminate both P_0 and P_1, which leaves

$$F_c - E_c = c\sqrt{h}\,\{F_h + h^{-1}(F_h - E_h)\},$$

a process much easier to *remember* than that of Art. 10.

To reduce the last, divide by $cF_c = c \,.\, (1+h)\,F_h$.

Then $$\frac{1-\aleph_c}{c} = \frac{\sqrt{h}}{1+h}\left\{1 + \frac{1-\aleph_h}{h}\right\} = \frac{c}{2}\left\{1 + \frac{1-\aleph_h}{h}\right\}$$ as in Art. 10.

This may please a learner as a confirmation

13. Legendre spared no pains to bring the series $P_0\ P_1\ P_2 \ldots$ into numerical valuation, and his processes are beautiful and instructive. But his main object was to form tables of *double* entry for $F(c\omega)$ and $E(c\omega)$. Wonderful was his toil and his apparent complete success, yet it would seem that unless one of the two given elements is the series 1, 2, 3, 4 &c. double entry is a mistake. Poisson coldly says of these great tables, that by these *in some sense* F and E can be

found. By the Higher Theory which Jacobi opened, we must hope to treat all by tables of *single entry*.

Since from a given c, we can deduce b, h and $g = \sqrt{(1-h^2)}$ by very simple arithmetic, to as many decimals as may be required, the student will readily understand that $c_1 b_1 c_2 b_2 \dots c_n b_n$ are all *attainable*, especially since n seldom need be made greater than 4 according to Legendre.

14. Thus far, by the method of multiple arcs, we can overcome the difficulties of accurately valuing the complete integrals F_c, E_c. To extend this to the indefinite integrals $F(c\omega)$, $E(c\omega)$, it is necessary to complete the series $P_0 P_0 P_1 P_2 \dots P_n \dots$. Legendre has executed this task, which is perhaps superseded in the modern treatment of these integrals. Nevertheless, his process is too beautiful and too instructive to omit.

If we had a linear equation connecting any three successive terms of the series $P_0 P_1 P_2 \dots P_n \dots$, then, knowing the two first, we could from them calculate them all, or by calculating any two distant terms of the series, we could work back from them to P_1 and P_0; and if P_0 were known already, this would furnish corroboration. It will also appear, that on going forward towards P_n small errors must be magnified; while in the process of working backwards from P_n and P_{n-1} all error becomes less and less.

15. Legendre by the following artifice obtains the linear equation desired.

In Art. 3 we have the development

$$\mu\Delta^{-1} = P_0 - 2P_1 \cos 2\omega + 2P_2 \cos 4\omega - 2P_3 \cos 6\omega + \&c.$$

From this we may deduce the development of Δ. Differentiate

$$\Delta^2 = 1 - c^2 \sin^2 \omega\,;$$

then $$2\Delta d\Delta = -2c^2 \sin\omega \cos\omega \, d\omega = -c^2 \sin 2\omega d\omega,$$

and $$d\Delta = -\tfrac{1}{2}c^2 \sin 2\omega \,.\, \Delta^{-1} \,.\, d\omega.$$

Multiply every term in the series for Δ^{-1} by $\sin 2\omega$, therefore

$$\mu \,.\, d\Delta = -\tfrac{1}{2}c^2 \{P_0 \sin 2\omega - 2P_1 \cos 2\omega \sin 2\omega + 2P_2 \cos 4\omega \sin 2\omega - \&c.\}.$$

Now for all values of r,

$$2 \cos 2r\omega \,.\, \sin 2\omega = \sin (2r+2)\,\omega - \sin (2r-2)\,\omega,$$

thus we can reduce the last series to linear sines. Also

$$\mu^{-1}.\tfrac{1}{2}c^2 = (1+h)\tfrac{1}{2}c^2 = c\sqrt{h},$$

so

$$d\Delta = -c\sqrt{h}\left[\begin{array}{l} P_0\sin 2\omega - P_1\sin 4\omega + P_2\sin 6\omega - P_3\sin 8\omega + \&c. \\ -P_2\sin 2\omega + P_3\sin 4\omega - P_4\sin 6\omega + P_5\sin 8\omega - \&c. \end{array}\right\} d\omega.$$

Integrate therefore

$$\Delta = P' + c\sqrt{h}\{(P_0 - P_2)\tfrac{1}{2}\cos 2\omega - (P_1 - P_3)\tfrac{1}{4}\cos 4\omega$$
$$+ (P_2 - P_4)\tfrac{1}{6}\cos 6\omega - (P_3 - P_5)\tfrac{1}{8}\cos 8\omega + \&c.\}.$$

The first term P' is the constant of integration. Since

$$\int_0 \Delta d\omega = E(c\omega),$$

we easily find that $E_c = P'.\tfrac{1}{2}\pi$, whence if E_c is counted as known the constant P' is determined.

16. Resume the equivalence

$$\Delta^2 = \mu^2(1 + 2h\cos 2\omega + h^2),$$

or $$(1+h)\Delta = \mu\Delta^{-1}(1 + 2h\cos 2\omega + h^2).$$

If in the last we replace Δ and Δ^{-1} by the series in linear cosines of multiples of ω, and again reduce to linear cosines on the right hand, we shall not produce an identical equation such as we might call a *Truism*. For P' is found in Δ, and *not* in Δ^{-1}. So the result will not be void, as $A = A$. If for conciseness we represent the result by

$$Z_0 - Z_1\cos 2\omega + Z_2\cos 4\omega \;\&c. + Z_n\cos 2n\omega \mp \&c. = 0,$$

the series Z is wholly free from ω. In this, the series P, containing h only, must be so related, that the last equation shall be true for all values of h and for all values of ω.

Nor only so, but if for $\cos 2\omega$ we restore $\tfrac{1}{2}(v + v^{-1})$, the equation will remain true for all *real* values of v.

Evidently, all these claims will be fulfilled if *every* $Z = 0$. In no other way does so difficult a result seem possible. It demands, in particular, $$(1+h)P' = (1+h^2)P_0 - 2hP_1;$$

and in general for any integer n,

$$(1 + \tfrac{1}{2}n^{-1})P_{n+1} = (h^{-1} + h)P_n - (1 - \tfrac{1}{2}n^{-1})P_{n-1} \quad \ldots\ldots(A),$$

17. This is just what we were seeking for; a linear equation connecting any three successive terms of the series P. But the multiplier $(h^{-1}+h)$ is vexatiously great when h is small, and accumulates any original error, if we move *towards* P_{n+1}.

In Art. 3 we had

$$P_n = k_n h^n + k_{n+1} k_1 h^{n+2} + k_{n+2} k_2 h^{n+4} + \&c.,$$

of which every term divides by h^n. Also k_{n+r} is a multiple of k_n.

If then we put $$P_n = k_n h^n Q_n,$$

we find more simply

$$\left.\begin{aligned} Q_n = 1 + \frac{2n+1}{2n+2} k_1 h^2 + \frac{2n+1}{2n+2} \cdot \frac{2n+3}{2n+4} k_2 h^4 \\ + \frac{2n+1}{2n+2} \cdot \frac{2n+3}{2n+4} \cdot \frac{2n+5}{2n+6} \cdot k_3 h^6 + \&c. \end{aligned}\right\}$$

Observe further, whatever the value of r,

$$\frac{2n+r-1}{2n+r} = \left(1 - \frac{1}{2n+r}\right),$$

which shows that the series admits the new form

$$Q_n = H_0 - \frac{H_1}{n+1} + \frac{H_2}{n+1\,.\,n+2} - \frac{H_3}{n+1\,.\,n+2\,.\,n+3} + \&c.\ldots(C),$$

in which the series H is free from n. To fix H_0, we have only to make n infinite, therefore $H_0 = Q$ with n infinite

$$= 1 + k_1 h^2 + k_2 h^4 + \&c. = (1-h^2)^{-\frac{1}{2}}.$$

Let $$h = \sin.\eta; \quad \therefore H_0 = \sec.\eta.$$

Next, we must deduce from equation (A) the relation between Q_{n+1}, Q_n, and Q_{n-1}. This is matter of algebraic routine; and with a little attention, you may give in the form

$$(Q_{n-1} - Q_n) = h^2(Q_n - Q_{n+1}) - (4n\,.\,n+1)^{-1}\,.\,h^2 Q_{n+1} \ldots\ldots(B).$$

18. From the last it is manifest, that if Q_{n+1} and Q_n be somehow attained, the multiplier h^2 on the right member reduces any error, so that we attain Q_{n-1} from this equation with advantage. But we have yet to find H_1, H_2, H_3 ... before equation (C) avails us.

In it first write $(n-1)$ for n, then you at once deduce

$$Q_{n-1} - Q_n = \frac{-1 \,.\, H_1}{n \,.\, n+1} + \frac{2H_2}{n \,.\, n+1 \,.\, n+2} - \frac{3H_3}{n \,.\, n+1 \,.\, n+2 \,.\, n+3} + \text{\&c.} \quad \ldots\ldots\text{(D)}.$$

In the last, we further change n to $n+1$, making the left hand $Q_n - Q_{n+1}$; but further on the right hand observe that

$$\frac{1}{n+1 \,.\, n+2} = \frac{(n+2)-2}{n \,.\, n+1 \,.\, n+2} = \frac{1}{n \,.\, n+1} - \frac{2}{n \,.\, n+1 \,.\, n+2};$$

$$\frac{1}{n+1 \,.\, n+2 \,.\, n+3} = \frac{(n+3)-3}{n \,.\, n+1 \,.\, n+2 \,.\, n+3} = \frac{1}{n \,.\, n+1 \,.\, n+2} - \frac{3}{n \,.\, n+1 \,.\, n+2 \,.\, n+3};$$

and so on, therefore

$$Q_n - Q_{n+1} = \frac{-1 \,.\, H_1}{n \,.\, n+1} + \frac{2H_2 + 1 \,.\, 2H_1}{n \,.\, n+1 \,.\, n+2} - \frac{3H_3 + 2 \,.\, 3H_2}{n \,.\, n+1 \,.\, n+2 \,.\, n+3} + \text{\&c.} \ldots\ldots\ldots\ldots\ldots\text{(E)}.$$

But also

$$\frac{Q_{n+1}}{4n \,.\, n+1} = \frac{H_0}{4n \,.\, n+1} - \frac{H_1}{4n \,.\, n+1 \,.\, n+2} + \frac{H_2}{4n \,.\, n+1 \,.\, n+2 \,.\, n+3} - \text{\&c.} \ldots\ldots\ldots\ldots\text{(F)}.$$

19. Into (B) above introduce the values of the Q series from (D), (E), (F), and compare the terms whose denominators are of the same degree in n; thence you find

$$(1-h^2)\, H_1 = \tfrac{1}{4} h^2 H_0;$$

and generally $\quad r \,.\, (1-h^2)\, H_r = (r - \tfrac{1}{2})^2 \,.\, h^2 \,.\, H_{r-1};$

whence, since $\quad \dfrac{h^2}{1-h^2} = \dfrac{\sin^2 \eta}{\cos^2 \eta} = \tan^2 . \eta,$

$$H_1 = \tfrac{1}{4} \tan^2 \eta \,.\, H_0;$$

and generally $\quad rH_r = \tfrac{1}{4}(r - \tfrac{1}{2})^2 \,.\, \tan^2 \eta \,.\, H_{r-1};$

so that

$$Q_n = \sec . \eta \left\{1 - \frac{1^2 . \tan^2 \eta}{2 . 2n+2} + \frac{1^2 . 3^2 . \tan^4 \eta}{2 . 4 . 2n+2 . 2n+4} \right.$$

$$\left. - \frac{1^2 . 3^2 . 5^2 . \tan^6 \eta}{2 . 4 . 6 . 2n+2 . 2n+4 . 2n+6} + \&c.\right\} \ldots (G).$$

The terms being of alternate sign are advantageous for calculation, and on our assumption that h is small, so is $\tan^2 \eta$. The larger n the better is the convergence; and if at the worst four terms must be computed, the process differs little for Q_n and Q_{n+1}. Knowing these we can work back towards Q_1 and Q, and can obtain verification from $Q_0 = P_0 = \frac{F_h}{\frac{1}{2}\pi}$. We have now also finally

$$F(c\omega) = (1+h)\{Q_0 . \omega - k_1 h Q_1 \sin 2\omega + \tfrac{1}{2} k_2 h^2 Q_2 \sin 4\omega$$

$$- \tfrac{1}{3} k_3 h^3 Q_3 \sin 6\omega + \&c.\},$$

and $E(c\omega)$ is more simply found by taking $\int_0 \Delta d\omega$ from the series above for Δ which has P' for its first term, Art. 15.

20. One more simplification seems to remain, since *every* Q has $\sec \eta$ as a common factor. Put $Q = \sec \eta . R$; so that

$$R_n = 1 - \frac{1^2 . \tan^2 \eta}{2 . 2n+2} + \frac{1^2 . 3^2 . \tan^4 \eta}{2 . 4 . 2n+2 . 2n+4}$$

$$- \frac{1^2 . 3^2 . 5^2 . \tan^6 \eta}{2 . 4 . 6 . 2n+2 . 2n+4 . 2n+6} + \&c.$$

Then equation (B) remains the same with R as with Q, so that

$$R_{n-1} - R_n = h^2 . \left\{(R_n - R_{n+1}) - \frac{R_{n+1}}{4n . n+1}\right\},$$

which R_{n-1} is deducible from R_n and R_{n+1}.

1ext, observe that

how at
error, so

$$(1 + h)\sec \eta = \frac{1+h}{\sqrt{(1-h^2)}} = \sqrt{\frac{1+h}{1-h}}$$

But we ha

$$= \sqrt{\frac{1+\sin \eta}{1-\sin \eta}} = \tan (45^\circ + \tfrac{1}{2}\eta).$$

Now if we have *given* γ as the arc of $c = \sin\gamma$, $b = \cos\gamma$, we have

$$h = \frac{1 - \cos\gamma}{1 + \cos\gamma}, \text{ or } \sin\eta = (\tan\tfrac{1}{2}\gamma)^2 \text{ to find } \eta.$$

There remains

$$F(c\omega) = \tan(45^\circ + \tfrac{1}{2}\eta)\{R_0\omega - k_1 hR_1 \cos 2\omega + \tfrac{1}{2}k_2 h^2 R_2 \cos 4\omega - \tfrac{1}{3}\,.\,\&c.\},$$

and if a table were to be formed with γ for argument, the values of hR, R_1, $R_2 \ldots$ would appear as far as needed.

Of course $E(c\omega)$ is in like manner simplified; but it is hardly worth while to go further here.

It will be observed that as the series R always tends to **1** as its limit, the series for $F(c\omega)$ barely converges on the scale of 1, h, h^2, $h^3 \ldots$, which, unless h is very small, is vastly inferior to that which is attained (as will be seen) by higher methods: moreover, since $\cos 2n\omega$ changes sign with the increase of n, we cannot account the terms of our series to be alternately positive and negative. Thus far we have attained only what in the early stages of the Integral Calculus would have seemed a real feat. Up to

$$c = \frac{3}{5}, \quad b = \frac{4}{5}, \quad h = \frac{5-4}{5+4} = \frac{1}{9},$$

the convergence is certainly respectable, and once would have seemed a good result. When

$$c = \frac{5}{13}, \quad b = \frac{12}{13}, \quad h = \frac{1}{25}.$$

CHAPTER III.

NEW AUXILIARIES.

1. TRIGONOMETRY is made more elegant and more intelligible by introducing many related functions beside the ancient Arc and Chord. In our more complex topic the same necessity of Auxiliar Functions presses still more strongly.

Let x be a circular arc, so related to the amplitude ω, that

$$\frac{x}{\frac{1}{2}\pi} = \frac{F(c,\ \omega)}{F(c,\ \frac{1}{2}\pi)}.$$

Then whenever $\omega = n.(\frac{1}{2}\pi)$, x also $= n(\frac{1}{2}\pi)$; i.e. x and ω coincide at the close of every quadrant; simply because

$$F(c,\ n.\tfrac{1}{2}\pi) = n.F(c,\ \tfrac{1}{2}\pi).$$

Thus x affords a sort of *average* of the Nome F. I call x the ***Mesonome.*** Also $F(c\omega) = Cx$, since $F_c = \frac{1}{2}\pi.C$.

I find it convenient to use the notation, $\omega = ({}_cx)$; meaning, "ω is the amplitude whose Mesonome is x with modulus c," *the modulus being written below the line, and to the left,* in direct opposition to an exponent. Then if a trigonometrical symbol *precede,* the modulus will not affect it; but $\sin({}_cx)$ will mean $\sin\omega$. When

$$c = 0,\ \left.\begin{aligned} F(c\omega) &= \omega \\ &= x \end{aligned}\right),$$

$$\therefore\quad ({}_0x) = \omega = x.$$

Also $\sin\omega$ may be replaced by $\sin({}_cx)$, upon occasion.

When amplitudes are supplementary, so are their mesonomes.

For, if $$\omega_1 = \pi - \omega, \quad \sin\omega_1 = \sin\omega.$$

Also $$d\omega_1 = -d\omega,$$

$$\therefore \quad dF\omega_1 = -dF\omega,$$

$$F\omega + F\omega_1 = const. = 2F_c.$$

Hence also $$\frac{x}{\frac{1}{2}\pi} + \frac{x_1}{\frac{1}{2}\pi} = 2,$$

or $$x + x_1 = \pi. \quad \text{Q.E.D.}$$

2. Since Δ is positive and < 1, but $\Delta^{-1} > 1$, F increases (between $\omega = 0$ and $\omega = \frac{1}{2}\pi$) faster than ω, and E slower. Thus, within these limits, if ψ is $> \omega$, $$\frac{F(\psi)}{E(\psi)} \text{ is } > \frac{F(\omega)}{E(\omega)},$$

so that also $$\frac{F_c}{E_c} > \frac{F(\omega)}{E(\omega)} \text{ or } \frac{E(\omega)}{E_c} > \frac{F(\omega)}{F_c},$$

and $$E(\omega) > \frac{E_c}{F_c} . F(\omega).$$

Thus $$[E(\omega) - \aleph_c . F(\omega)]$$

is positive in the first quadrant. I call this combination the *Nomiscus.*

When ω reaches $\frac{1}{2}\pi$, the Nomiscus *vanishes:* so does it at the end of every quadrant. It is evidently positive in the odd quadrants, negative in the even. I denote it by $G(c\omega)$. Then

$$G(\omega) = E(\omega) - \aleph_c . F(\omega)$$

and $E(\omega)$ is knowable from

$$G(\omega) + \aleph_c . F(\omega).$$

3 The Nomiscus *fluctuates* like a sine, between narrow limits. Also when $c = zero$, it vanishes; for then

$$\aleph_c = 1, \text{ and } E = F.$$

Indeed when c^2 *is very small,* $\aleph_c = 1 - \frac{1}{2}c^2$;

$$\therefore \Delta^2 - \aleph_c = \frac{1}{2}c^2 - c^2\sin^2\omega = \frac{1}{2}c^2(1 - 2\sin^2\omega)$$

and $$\Delta^{-1} = 1 + \frac{1}{2}c^2\sin^2\omega;$$

from which

$$G(\omega) = \int (\Delta^2 - \aleph_c) . \Delta^{-1} d\omega = \frac{1}{2}c^2 \int_0 (\cos 2\omega)(1 + \frac{1}{2}c^2\sin^2\omega)\, d\omega.$$

But c^4 being omissible, this gives

$$G(\omega) = \tfrac{1}{4}c^2 \,.\, \sin 2\omega,$$

by integration.

Evidently $$G(-\omega) = -G(\omega),$$

or the function is *odd*, like F and E. Since $G(n \,.\, \tfrac{1}{2}\pi) = 0$, we easily infer that $G(\pi - \omega)$ and $G(n\pi - \omega) = -G(\omega)$.

4. In the treatment of Π, the third integral, its auxiliary

$$\daleth(c\omega) = \int_0 G(c\omega)\, dF(c\omega)$$

will be needed; and it is expedient to be familiar with it at once. I call it the *Diplonome.* Evidently

$$\daleth(-\omega) = \int_0 G(-\omega)\, d(-F) = \int_0 G dF = \daleth(\omega),$$

or the function is *even.* Since in the second quadrant G becomes negative, $\daleth$ rises to a maximum when $\omega = \tfrac{1}{2}\pi$. We may expect it to vanish when $\omega = \pi$.

In fact

$$\daleth(\pi - \omega) = \int_0 G(\pi - \omega)\, dF(\pi - \omega) = \int_0 - G\omega \,.\, d(-F\omega)$$

$$= \int_0 + G\omega \,.\, dF\omega = \daleth\omega.$$

Make ω vanish; $\daleth\omega$ vanishes also; $\therefore \daleth(\pi) = 0$. So in general

$$\daleth(n\pi) = 0;$$

whence also $$\daleth(n\pi \pm \omega) = \daleth\omega.$$

This function being always positive, is comparable to $\sin^2\omega$ in periodicity. If c^2 is small, and $G = \tfrac{1}{4}c^2 \sin 2\omega_1$,

$$dF = (1 + \tfrac{1}{2}c^2 \sin^2\omega)\, d\omega,$$

we may omit the term multiplied by c^4, and find

$$\daleth\omega = \int_0 \tfrac{1}{4}c^2 \,.\, \sin 2\omega d\omega = \tfrac{1}{8}c^2 \,.\, (1 - \cos 2\omega).$$

5. *Conjugate Nomes.* In Chapter I., Art. 10, with $\tan\omega = t$, we had

$$dF = \frac{dt}{\sqrt{(1+b^2t^2)}\sqrt{(1+t^2)}}.$$

Let $btu = 1$, and substitute $(bu)^{-1}$ for t in this equation,

$$\therefore\ dF(c\omega) = \frac{d\,(bu)^{-1}}{\sqrt{(1+b^2b^{-2}u^{-2})}\sqrt{(1+b^{-2}u^{-2})}} = \frac{-du}{\sqrt{(u^2+1)}\sqrt{(b^2u^2+1)}}.$$

Let $u = \tan\theta$, and the last fraction visibly is $-dF(c,\ \theta)$, so that

$$dF(c\omega) = -dF(c\theta);$$

or integrated, $$F(c\omega) + F(c\theta) = const.$$

If ω increase from 0 to $\frac{1}{2}\pi$, t increases from 0 to ∞, u decreases from ∞ to 0, and θ decreases from $\frac{1}{2}\pi$ to 0. Hence to make $\omega = 0$ makes $\theta = \frac{1}{2}\pi$; therefore the *constant* is F_c.

Thus the *trigonometrical* relation $b\,.\tan\omega\,.\tan\theta = 1$ is coincident with the *transcendental* relation

$$F(c\omega) + F(c\theta) = F_c.$$

In this case the Amplitudes are called *Conjugate.*

COR. The Mesonomes are then *Complementary.* For the relation

$$\frac{F(c\omega)}{F_c} + \frac{F(c\theta)}{F_c} = 1,$$

gives us (if x' is Mesonome to θ)

$$\frac{x}{\frac{1}{2}\pi} + \frac{x'}{\frac{1}{2}\pi} = 1,$$

or $$x + x' = \tfrac{1}{2}\pi.$$

It is convenient to write ω and ω^0 for Conjugate amplitudes, the *zero* being equivalent to a *modifying accent* or index, *not an exponent.* Observe this *in all that follows.*

From the primary equation $b\tan\omega\tan\omega^0 = 1$, or $\cot\omega^0 = b\tan\omega$, three other relations follow, with which the student ought to be quite familiar. Easy trigonometry deduces them.

$$\text{(a)} \qquad \sin \omega^0 = \frac{\cos \omega}{\Delta};$$

$$\text{(b)} \qquad \cos \omega^0 = \frac{b \sin \omega}{\Delta};$$

$$\text{(c)} \qquad \Delta^0 = \frac{b}{\Delta}.$$

In Trigonometry, the equation $\tan \psi = b \tan \omega$, is developed

$$\left(\text{when } h = \frac{1-b}{1+b}\right)$$

into $$\psi = \omega - h \sin 2\omega + \tfrac{1}{2}h^2 \sin 4\omega - \tfrac{1}{3}h^3 \sin 6\omega + \&\text{c}.$$

Here therefore, since $\cot \omega^0 = \tan (\tfrac{1}{2}\pi - \omega^0)$,

putting $$\psi = \tfrac{1}{2}\pi - \omega^0,$$

you have $$\tan \psi = b \tan \omega,$$

$$\therefore \quad (\tfrac{1}{2}\pi - \omega^0) = \omega - h \sin 2\omega + \tfrac{1}{2}h^2 \sin 2\omega - \tfrac{1}{3}\,.\,\&\text{c}....$$

If x' be the Mesonome to $F(c\omega^0)$ the equation

$$F + F^0 = F_c$$

yields $$Cx + Cx' = \tfrac{1}{2}\pi C,$$

hence $$x + x' = \tfrac{1}{2}\pi.$$

Hence when the amplitudes are *conjugate*, the mesonomes are *complementary.*

6. *Conjugate Epinomes.* We retain the same relation of the amplitudes, $$\therefore \quad \frac{d\omega}{\Delta} + \frac{d\omega^0}{\Delta^0} = 0.$$

Multiply by $$\Delta^2 = \frac{b^2}{\Delta^{02}},$$

$$\therefore \quad \Delta d\omega + \frac{b^2 d\omega^0}{\Delta^{03}} = 0.$$

By the table in Chapter I., Art. 4, we know $\int_0 \frac{b^2 d\omega}{\Delta^3}$ and we may replace ω by ω^0. Integrating then

$$\int \Delta d\omega + \left[E^0 - c^2 \frac{\sin \omega^0 \cos \omega^0}{\Delta^0}\right] = \textit{const.}$$

Since $$\sin\omega^0 = \frac{\cos\omega}{\Delta},$$

by symmetry $$\sin\omega = \frac{\cos\omega^0}{\Delta^0}.$$

Hence $$E + E^0 - c^2 \sin\omega^0 \sin\omega = const.$$

Make $$\omega = \tfrac{1}{2}\pi, \quad \omega^0 = 0, \quad \therefore\ const. = E_c;$$

or $$E + E^0 - E_c = c^2 \sin\omega \sin\omega^0.$$

7. *Conjugate Nomisci.* If ϕ is a function composed of $E - aF$, where a is constant,

$$\phi + \phi^0 - \phi_c = (E - aF) + (E^0 - aF^0) - (E_c - aF_c).$$

But $$F + F^0 - F_c = 0 \text{ and } E + E^0 - E_c = c^2 \sin\omega \sin\omega^0,$$

$$\therefore\ \phi + \phi^0 - \phi_c = c^2 \sin\omega \sin\omega^0.$$

Let the arbitrary constant a be $\aleph_c$. Then ϕ becomes the Nomiscus G, and $$\phi_c = G_c = \text{zero}.$$

Hence we have $$G + G^0 = c^2 \sin\omega \,.\, \sin\omega^0.$$

8. *Conjugate Diplonomes.* Multiply the last equation by

$$dF = -\,dF^0 = \frac{d\omega}{\Delta},$$

then

$$GdF - G^0 dF^0 = \frac{c^2 \sin\omega \,.\, \sin\omega^0 d\omega}{\Delta} = \frac{c^2 \sin\omega \cos\omega\, d\omega}{\Delta^2} = \frac{\tfrac{1}{2}c^2 \,.\, d\,(\sin^2\omega)}{1 - c^2 \sin^2\omega}.$$

Integrate; then

$$\daleth(\omega) - \daleth(\omega^0) + const. = -\tfrac{1}{2}\log(\Delta^2) = -\log\Delta.$$

Let $$\omega^0 = \tfrac{1}{2}\pi, \quad \omega = 0, \quad \Delta = 1,$$

$$\therefore\ const. = \daleth_c,$$

so that $$\daleth\omega - \daleth\omega^0 + \daleth_c = -\log\Delta\,(c\omega).$$

Next, make $$\omega = \omega^0, \quad \cot^2\omega = b, \quad \sin^2\omega = \frac{1}{1+b};$$

$$\therefore\ \Delta^2 \text{ becomes } = 1 - \frac{c^2}{1+b} = 1 - (1-b) = b.$$

Finally $$2\daleth_c = -\log b.$$

9. *Differential relation of the Mesonome to ρ, the Promodulus.* A particular differential equation *with ω constant* deserves notice.

In Chapter I., Art. 11, we found

$$b^2 \frac{d\,(cF)}{dc} = E - c^2 \frac{\sin\omega \cos\omega}{\Delta},$$

or

$$b^2 . F + b^2 c \frac{dF}{dc} = E - c^2 \sin\omega \sin\omega^0.$$

Make

$$\omega = \tfrac{1}{2}\pi, \quad \omega^0 = 0,$$

$$\therefore \; b^2 F_c + b^2 c . \frac{dF_c}{dc} = E_c.$$

Multiply the last by $\dfrac{F}{F_c}$ and subtract from preceding, whence

$$b^2 c \left\{\frac{dF}{dc} - \frac{F}{F_c} \cdot \frac{dF_c}{dc}\right\} \text{ or } b^2 c . \frac{F_c dF - F dF_c}{F_c . dc}$$

$$= E - E_c . \frac{F}{F_c} - c^2 \sin\omega \sin\omega^0 = G - c^2 \sin\omega \sin\omega^0 = -G^0.$$

Now $\quad \dfrac{d}{dc}\left(\dfrac{x}{\frac{1}{2}\pi}\right) = \dfrac{d}{dc} \cdot \dfrac{F}{F_c} = \dfrac{F_c dF - F dF_c}{(F_c)^2 dc}$

which is now $= \dfrac{-G^0}{b^2 c . F_c} = \dfrac{-G^0}{\frac{1}{2}\pi C . b^2 c}$.

Multiply by $\frac{1}{2}\pi$.

Eliminate dc by $dc = -b^2 c . C^2 d\rho$. Then with ω constant and c variable,

$$dx = \frac{-G^0}{C} \cdot \frac{dc}{b^2 c} = \frac{G^0}{C} . C^2 d\rho,$$

or

$$\frac{dx}{d\rho} = C . G^0;$$

a remarkably simple formula

*** *Imaginary Amplitudes.*

10. We obtained the relation $F + F^0 = F_c$, by making $t = \tan\omega$,

$$F = \int_0 \frac{dt}{\sqrt{(1+t^2)}\sqrt{(1+b^2t^2)}}.$$

Now if $v = \sin\omega$, we get

$$F = \int_0 \frac{dv}{\sqrt{(1-v^2)}\sqrt{(1-c^2v^2)}},$$

which has a remarkable analogy to the other form. In trying to imitate the former process, we involve ourselves in imaginatries.

Comparing t^2 with $-v^2$, and b^2 with c^2, the thought occurs to assume
$$\sin\omega = \sqrt{-1}\tan\psi,$$
$$\therefore\ \sin^2\omega = -\tan^2\psi,$$
$$\cos^2\omega = \sec^2\psi,$$
$$\Delta(c\omega) = 1 + c^2\tan^2\psi = 1 + (1-b^2)\tan^2\psi = \sec^2\psi - b^2\tan^2\psi$$
$$= \sec\psi\,.\,\Delta(b,\psi),$$
also
$$\Delta(b\psi) = \sec\omega\,.\,\Delta(c\omega).$$
Hence
$$dF(c\omega) = \frac{d\omega}{\Delta(c\omega)} = \frac{d\sin\omega}{\cos\omega\Delta(c\omega)}$$
$$= \frac{\sqrt{-1}\,.\,d\tan\psi}{\sec^2\psi\,.\,\Delta(b\psi)} = \frac{\sqrt{-1}\,.\,d\psi}{\Delta(b\psi)} = \sqrt{-1}\,.\,dF(b\psi).$$
Integrate, observing that ω ψ vanish together.

Then
$$F(c\omega) = \sqrt{-1}\,.\,F(b\psi).$$
I call $F(c\omega)$ and $F(b\psi)$ *Anti-Nomes*. Let x and y be the Meso-nomes.
$$\therefore\ Cx = \sqrt{-1}\,.\,By.$$

11. The same transformation may be applied to $EG\daleth$, but new functions arise.

First, we have
$$E(c\omega) = \int_0 \Delta d\omega = \int_0 \Delta^2\,.\,dF = \int_0 \sec^2\psi\Delta^2(b\psi)\,.\,\sqrt{-1}\,.\,dF(b\psi)$$
$$= \sqrt{-1}\,.\int_0 \sec^2\psi\,.\,\Delta(b\psi)\,d\psi.$$
But by Table, Chapter I., Art. 4,
$$\int \sec^2\omega\,.\,\Delta d\omega = \tan\omega\Delta + F - E.$$

Therefore here
$$E(c\omega) = \sqrt{-1}\,\{\tan\psi\,.\,\Delta(b,\psi) + F(b,\psi) - E(b\psi)\}.$$
Hereby $G(c\omega)$ or
$$E - \aleph_c\,.\,F = \sqrt{-1}\,\{\tan\psi\,.\,\Delta(b\psi) - E(b,\psi) + (1-\aleph_c)F(b,\psi)\}.$$
This suggests a new Auxiliary $J(b\psi)$ to mean
$$E(b\psi) - (1-\aleph_c)F(b\psi),$$
which implies
$$J(c\omega) = E(c\omega) - (1-\aleph_b)F(c\omega).$$

Observe also that $\quad 1 - \aleph_b = \aleph_c - (\frac{1}{2}\pi BC)^{-1}$,

so that $\quad J(c\omega) = E(c\omega) - \left\{\aleph_c - \dfrac{1}{C \, . \, F_b}\right\} F(c\omega) = G(c\omega) + \dfrac{x}{F_b}.$

Thus $J(c, \omega)$ perpetually increases with x, and when $\omega = n \, . \, \frac{1}{2}\pi$,

$$J = \frac{n \, . \, \frac{1}{2}\pi}{F_b} = \frac{n}{B}.$$

In particular
$$J_c = \frac{1}{B},$$

$\therefore \quad J(c, n \, . \, \frac{1}{2}\pi) = n \, . \, J_c$, in entire analogy to F and E.

I call J the Antinomiscus.

12. We found that when c^4 is omissible $G(c\omega) = \frac{1}{4}c^2 \sin 2\omega$, and vanishes with c^2. But $J(c\omega)$ does not vanish with b^2, and it will presently concern us to estimate it when b^4 is omissible, but not b^2.

In Ch. I., Art. 16, we found

$$E = (1 - \tfrac{1}{2}b^2) \sin \omega + \tfrac{1}{2}b^2 \, . \int_0 \sec \omega \, . \, d\omega \text{...........(M)};$$

also, when b^4 is small enough to neglect,

$$b^2 \quad F = b^2 \int_0 \sec \omega \, d\omega \text{ ....(N)}.$$

We have just introduced $J(c\omega)$ for

$$E(c\omega) - (1 - \aleph_b) F(c\omega),$$

or since now
$$\aleph_b = 1 - \tfrac{1}{2}b^2,$$

omitting b^4 &c.,
$$J = E - \tfrac{1}{2}b^2 F,$$

or by giving to E and b^2F the values in M and N,

$$J = (1 - \tfrac{1}{2}b^2) \sin \omega.$$

13. Further, since

$$J(c\omega) = G(c\omega) + \frac{x}{F_b},$$

change ω to the conjugate ω^0, x to $\frac{1}{2}\pi - x$, then

$$J(c\omega^0) = G(c\omega^0) + \frac{\frac{1}{2}\pi - x}{F_b}.$$

Add these, observing that

$$G(c\omega) + G(c\omega^0) = c^2 \sin \omega \sin \omega^0 :$$

thence we find

$$J(c\omega)+J(c\omega^0)=c^2\sin\omega\sin\omega^0+\frac{\frac{1}{2}\pi}{F_b},$$

for the Conjugates; or again, since

$$\frac{\frac{1}{2}\pi}{F_b}=J_c,$$

$$J(c\omega)+J(c\omega^0)-J_c=c^2\sin\omega\sin\omega^0.$$

14. For convenient transformation we have now

$$G(c\omega)+\sqrt{-1}\,.\,J(b\psi)=\sqrt{-1}\,.\,\Delta(b\psi)\tan\psi,$$

or again, changing $cb\omega$ to $bc\psi$, which changes ψ to $-\omega$;

$$J(c\omega)+\sqrt{-1}\,.\,G(b\psi)=\Delta(c\omega)\tan\omega.$$

If y be mesonome to $b\psi$, it will be an *Anti-Mesonome* to x; and the equation

$$\{J(c\omega)-G(c\omega)\}-\sqrt{-1}\,\{J(b,\psi)-G(b\psi)\}=0$$

is equivalent to

$$\frac{x}{F_b}-\sqrt{-1}\,.\,\frac{y}{F_c}=0,\quad\text{or}\quad Cx=\sqrt{-1}\,By,$$

merely another form of

$$F(c\omega)=\sqrt{-1}\,.\,F(b\psi).$$

Introducing $$\rho=\tfrac{1}{2}\pi\,.\,\frac{B}{C},$$

we have $$\tfrac{1}{2}\pi\,.\,x=\sqrt{-1}\,.\,\rho\,.\,y,$$

which show an analogy of

$$\sqrt{-1}\,.\,\rho\quad\text{to}\quad\tfrac{1}{2}\pi.$$

15. *Antidiplonome.* Multiply

$$G(c\omega)+\sqrt{-1}\,J\,.\,(b\psi)=\Delta(c\omega)\,.\tan\omega$$

$$\text{by}\quad dF(c\omega)=\sqrt{-1}\,.\,dF(b\psi)=\frac{d\omega}{\Delta(c\omega)},$$

then $$d\daleth(c\omega)-J(b\psi)\,dF(b\psi)=\tan\omega\,.\,d\omega,$$

or $$\daleth(c\omega)-\int_0 J(b\psi)\,dF(b\psi)=-\log\cos\omega.$$

Write as a new Auxiliary

$$K(c\omega) = \int_0 J(c\omega)\, dF(c\omega).$$

Then $\qquad \daleth(c\omega) - K(b\psi) = -\log\cos\omega = \log\cos\psi.$

Of the auxiliary K I find only temporary need. But since

$$J(c\omega) = G(c\omega) + \frac{x}{F_b},$$

we have $\qquad K(c\omega) = \int_0 \left\{G(c\omega) + \frac{x}{F_b}\right\} dF(c\omega);$

and $\qquad F(c\omega) = Cx,$

we deduce $\qquad K(c\omega) = \daleth(c\omega) + \dfrac{\frac{1}{2}x^2\, C}{\frac{1}{2}\pi . B} = \daleth(c\omega) + \dfrac{x^2}{2\rho}.$

Let ρ' be to b, what ρ is to c, therefore

$$\rho' = \frac{\frac{1}{2}\pi C}{B},$$

hence $\qquad \rho\rho' = (\tfrac{1}{2}\pi)^2,$

and of $\rho\rho'$ one or other is $> \frac{1}{2}\pi$. Also

$$K(b\psi) = \daleth(b\psi) + \frac{y^2}{2\rho'}.$$

But from above

$$\daleth(c\omega) = K(b\psi) + \log\cos\psi = \daleth(b\psi) + \frac{y^2}{2\rho'} + \log\cos\psi.$$

I call $K(b\psi)$ the *Antidiplonome.* Also

$$(\tfrac{1}{2}\pi x)^2 = -\rho^2 y^2, \text{ or } \rho\rho' x^2 = -\rho^2 y^2, \text{ or } \frac{x^2}{\rho} = \frac{-y^2}{\rho'}.$$

Thus with one more slight change

$$\daleth(c\omega) = \daleth(b\psi) - \frac{x^2}{2\rho} - \log\cos\omega.$$

16. *Imaginary Periods.* Write $d\phi v$ for

$$\{(1 - v^2)(1 - c^2 v^2)\}^{-\frac{1}{2}}\, dv,$$

which gives $\qquad d\phi(v) = dF\omega,$ when $\omega = \sin v.$

When v increases from 1 to c^{-1}, the surd becomes imaginary, and in the interval, for $\sqrt{(1 - v^2)}$ we may better write

$$\pm\sqrt{-1}\ \sqrt{(v^2 - 1)},$$

but when v exceeds c^{-1}, the value of the surd is again real.

There is no discontinuity at either crisis. In *proof*, first let $v = 1 + y$ and y be infinitesimal; therefore

$$1 - v^2 = -2y,\ 1 - c^2v^2 = b^2,\ d\phi(v) = \frac{dy}{b\sqrt{(-2y)}} = \frac{\pm\sqrt{-2}dy}{2b\sqrt{(y)}},$$

and
$$\phi(v) = c' \pm \frac{\sqrt{-2}}{b} \cdot \sqrt{y},$$

thus the imaginary part vanishes with y, and there is no discontinuity. A like proof applies at $v = c^{-1}$.

Assume then

$$v^{-2} = 1 - b^2z^2,\ \therefore\ d\phi(cv) = \frac{\pm\sqrt{-1}\,.\,dz}{\sqrt{(1-z^2)}\sqrt{(1-b^2z^2)}}.$$

Integrate; then $\phi(cv) = \phi(c, 1) \pm \sqrt{-1}\ \phi(bz)$,

for when $z = 0$, $v = 1$, which fixes the constant of integration. Put $v = c^{-1}$, $z = 1$, and the imaginary part is complete; giving

$$\phi(c,\ c^{-1}) = F_c \pm \sqrt{-1}\,.\,F_b.$$

Thus between $v = 1$ and $v = c^{-1}$, the integral $\pm\sqrt{-1}\,.\,F_b$ is generated.

When v exceeds c^{-1}, let $cv = u^{-1}$, therefore

$$d\phi(v) = \frac{d(c^{-1}u^{-1})}{\sqrt{(1-c^{-2}u^{-2})}\sqrt{(1-u^{-2})}} = \frac{-du}{\sqrt{(c^2u^2-1)}\sqrt{(u^2-1)}}$$

$$= \frac{-du}{\sqrt{(1-c^2u^2)}\sqrt{(1-u^2)}} = -d\phi(u);$$

$$\therefore\ \phi(v) + \phi(u) = const.$$

When $u = 0$, $v = \infty$, $\therefore\ const. = \phi(\infty)$.

Also when $v = c^{-1}$, $u = 1$, or $const. = \phi(c^{-1}) + \phi(1)$.
Thus the integral generated from $v = 1$ to $v = \infty$, $= \phi(1)$ equal to that generated from $v = 0$ to $v = 1$. Also

$$\phi(\infty) = \phi(c^{-1}) + \phi(1) = F_c \pm \sqrt{-1}\ F_b + F_c = 2F_c \pm \sqrt{-1}\,.\,F_b.$$

Finally $\phi(v) + \phi(u) = \phi(\infty)$, when $cvu = 1$.

This is in close analogy to Chap. III., Art. 5, where

$$\psi(bt) + \psi(bt') = \psi(b\infty),\ \text{when } btt' = 1;$$

and we are forced to regard $\phi(cv)$ and $\phi(cu)$ as Imaginary Conjugates.

17. *Double Periodicity.* This was first enunciated by Abel. The notation suggested in Art. 1 may be here recalled. Write $\omega = |_c(x)$, *for*, "ω is the *amplitude* whose *mesonome* is x, when the *modulus* is c." Then $\sin |_c(x)$ means the sine of *this amplitude*, or is equivalent to $\sin \omega$. But when

$$c = 0,\ F(c\omega) = \omega,\ F_c = \tfrac{1}{2}\pi,$$

$$\therefore\quad x = \omega, \text{ and } \sin |_0(x) = \sin \omega = \sin x.$$

Thus, when $c = 0$, the subscript 0 is omissible. Also to change x to $\pi - x$ changes ω to $\pi - \omega$ (Art. 1), therefore

$$\sin |_c(\pi - x) = \sin |_c(x).$$

Now if in the last Art. xx' are the mesonomes corresponding to c, v and c, u, $\qquad \phi(v) = F(c\omega) = Cx,$

and similarly $\qquad \phi(u) = Cx',$

also $\qquad F_b = \rho C.$

The equation $cvu = 1$ is now written

$$c \,.\, \sin|_c(x) \,.\, \sin|_c(x') = 1,$$

and from $\quad \phi(v) + \phi(u) = 2F_c \pm \sqrt{-1} \,.\, F_b = \pi C \pm \sqrt{-1} \,.\, \rho C,$

we get $\qquad x + x' = \pi \pm \rho\sqrt{-1}.$

From this and $\qquad c \,.\, \sin|_c(x) \,.\, \sin|_c(x') = 1,$

eliminate x', and the result must be an *identical* equation, viz.

$$c \,.\, \sin|_c(x) \sin|_c(\pi \pm \rho\sqrt{-1} - x) = 1.$$

First change x into $\pi - x$, and $\pi - x$ into x, with

$$\sin|_c(\pi - x) = \sin|_c(x);\quad \therefore\ c \sin|_c(x) \,.\, \sin|_c(x \pm \rho\sqrt{-1}) = 1.$$

In the last, change x to $x \pm \rho\sqrt{-1}$; then

$$c \sin|_c(x \pm \rho\sqrt{-1}) \,.\, \sin|_c(x \pm 2\rho\sqrt{-1}) = 1.$$

Compare these two equations and you see that

$$\sin|_c(x \pm 2\rho\sqrt{-1}) = \sin|_c(x).$$

Thus to *add* $2\rho\sqrt{-1}$ to the *mesonome* (or to take it away) does not affect the *sine of the amplitude*. It readily follows that

$$\sin|_c(2m\pi \pm 2n\rho\sqrt{-1} + x) = \sin|_c(x),$$

indicating a real period 2π, and $2\rho\sqrt{-1}$ an imaginary period.

CHAPTER IV.

EULER'S INTEGRALS, AS IMPROVED BY LAGRANGE.

1. PROBLEM. Given the equation $F(\omega) + F(\theta) = F(\eta)$ to find an algebraically intelligible relation of $\omega\theta\eta$, with common modulus c.

Make θ constant, therefore

$$\frac{d\omega}{\Delta(\omega)} = \frac{d\eta}{\Delta(\eta)}.$$

Call each fraction du, and treat u as the leading variable.

Write $\omega', \eta', \omega'', \eta''$ for

$$\frac{d\omega}{du}, \frac{d\eta}{du}, \frac{d^2\omega}{du^2}, \frac{d^2\eta}{du^2}.$$

Then $\quad \omega' = \Delta(\omega),\ \eta' = \Delta(\eta),\ \text{or} \begin{cases} \omega'^2 = 1 - c^2 \sin^2\omega, \\ \eta'^2 = 1 - c^2 \sin^2\eta. \end{cases}$

Differentiating again

$$2\omega'' = -2c^2 \sin\omega \cos\omega = -c^2 \sin 2\omega; \quad \text{and} \quad 2\eta'' = -c^2 \sin 2\eta.$$

Let $\quad 2\omega = r - s, \quad 2\eta = r + s;$

$$\therefore \quad r = \eta + \omega, \quad s = \eta - \omega;$$

$$r's' = (\eta' + \omega')(\eta' - \omega') = \eta'^2 - \omega'^2 = -c^2(\sin^2\eta - \sin^2\omega),$$

so $\quad 2r's' = c^2(\cos 2\eta - \cos 2\omega) = c^2(\cos\overline{r+s} - \cos\overline{r-s})$

$$= -2c^2 . \sin r . \sin s.$$

Again

$$r'' = \eta'' + \omega'' = -\tfrac{1}{2}c^2(\sin 2\eta + \sin 2\omega) = -\tfrac{1}{2}c^2(\sin\overline{r+s} + \sin\overline{r-s})$$

$$= -c^2 \sin r . \cos s;$$

$$s'' = \eta'' - \omega'' = -\tfrac{1}{2}c^2(\sin\overline{r+s} - \sin\overline{r-s}) = -c^2 \cos r . \sin s.$$

Hence

$$\frac{r''}{r's'} = \frac{-c^2 \sin r \cos . s}{-c^2 \sin r \sin s} = \frac{\cos s}{\sin s}; \quad \frac{s''}{r's'} = \frac{-c^2 \cos r . \sin s}{-c^2 . \sin r . \sin s} = \frac{\cos r}{\sin r};$$

otherwise $$\frac{r''}{r'} = \frac{\cos s . s'}{\sin s}; \quad \frac{s''}{s'} = \frac{\cos r . r'}{\sin r}.$$

Multiply by du; therefore

$$\frac{dr'}{r'} = \frac{d \sin s}{\sin s}; \quad \frac{ds'}{s'} = \frac{d \sin r}{\sin r}.$$

Integrate; therefore

$$\log r' = \log \alpha + \log \sin s; \quad \log s' = \log \beta + \log \sin r;$$

or $$r' = \alpha \sin s; \quad s' = \beta \sin r.$$

2. Further $$\frac{dr}{ds} = \frac{r'}{s'} = \frac{\alpha \sin s}{\beta \sin r};$$

whence $$\alpha \sin s . ds = \beta . \sin r . dr;$$

which also is integrable. Then

$$\alpha \cos s - \beta \cos r = \gamma;$$

thus we have three constants of integration α, β, γ. Now

$$r' = \eta' + \omega' = \Delta\eta + \Delta\omega, \quad s' = \eta' - \omega' = \Delta\eta - \Delta\omega;$$

$$\therefore \quad \frac{\Delta\eta + \Delta\omega}{\Delta\eta - \Delta\omega} = \frac{r'}{s'} = \frac{\alpha \sin s}{\beta \sin r} = \frac{\alpha \sin (\eta - \omega)}{\beta \sin (\eta + \omega)}.$$

But when $\omega = 0$, the original equation, our hypothesis, makes

$$\eta = \theta, \therefore \frac{\Delta\theta + 1}{\Delta\theta - 1} = \frac{\alpha}{\beta},$$

which is therefore negative. The ratio of α to β being alone concerned, assume

$$\alpha = 1 + \Delta(\theta), \quad -\beta = 1 - \Delta(\theta).$$

In $$\gamma = \alpha \cos s - \beta \cos r,$$

make $$\omega = 0, \quad \eta = \theta, \quad s = \eta = \theta, \quad r = \eta = \theta;$$

$$\therefore \quad \gamma = \alpha \cos \theta - \beta \cos \theta = (\alpha - \beta) \cos \theta = 2 \cos \theta.$$

Thus we have obtained as integral of

$$\frac{d\omega}{\Delta\omega} = \frac{d\eta}{\Delta\eta},$$

two forms, which must be identical; viz.

$$\frac{\Delta\eta + \Delta\omega}{\Delta\eta - \Delta\omega} = \frac{\Delta\theta + 1}{\Delta\theta - 1} \cdot \frac{\sin(\eta - \omega)}{\sin(\eta + \omega)},$$

and $$(1 + \Delta\theta)\cos(\eta - \omega) + (1 - \Delta\theta)\cos(\eta + \omega) = 2\cos\theta.$$

But the latter by expanding $\cos(\eta \mp \omega)$ becomes

$$\cos\eta\cos\omega + \sin\eta \,.\, \sin\omega \,.\, \Delta\theta = \cos\theta.$$

But ω and θ are exchangeable in the original;

$$\therefore\ \cos\eta\cos\theta + \sin\eta\sin\theta\Delta\omega = \cos\omega.$$

Again in 1st to change ω to $-\eta$, and η to $-\omega$, reproduces our first equation,

$$\therefore\ \cos\omega\cos\theta - \sin\omega\sin\theta\Delta\eta = \cos\eta.$$

Thus we obtain 3 equations out of one.

If by involution we remove the Cardinal Surd, any one of the three equations yields (A)

$$\tfrac{1}{2}(\sin^2\omega + \sin^2\theta + \sin^2\eta) + \cos\omega\cos\theta\cos\eta = 1 + \tfrac{1}{2}c^2\sin^2\omega \,.\, \sin^2\theta \,.\, \sin^2\eta.$$

The *first* integral is reducible to

$$\sqrt{\frac{1 - \Delta\eta}{1 + \Delta\eta}} = \frac{c\sin(\omega + \theta)}{\Delta\omega + \Delta\theta}.$$

In proof, first multiply by -1; therefore

$$\frac{\Delta\omega + \Delta\eta}{\Delta\omega - \Delta\eta} = \frac{1 + \Delta\theta}{1 - \Delta\theta} \cdot \frac{\sin(\eta - \omega)}{\sin(\eta + \omega)}.$$

But $$\frac{\Delta\omega + \Delta\eta}{\Delta\omega - \Delta\eta} = \frac{(\Delta\omega + \Delta\eta)^2}{\Delta^2\omega - \Delta^2\eta},$$

of which the denominator

$$= c^2(\sin^2\eta - \sin^2\omega) = c^2\sin(\eta - \omega)(\sin\eta + \omega);$$

$$\therefore\ \frac{(\Delta\omega + \Delta\eta)^2}{c^2\sin(\eta - \omega)\sin(\eta + \omega)} = \frac{(1 + \Delta\theta)}{1 - \Delta\theta} \cdot \frac{\sin(\eta - \omega)}{\sin(\eta + \omega)},$$

whence $$\frac{\Delta\omega + \Delta\eta}{c\sin(\eta - \omega)} = \pm\sqrt{\frac{1 + \Delta\theta}{1 - \Delta\theta}}.$$

Change $\eta\theta$ to $-\theta$, $-\eta$, since

$$F(-\theta) = F(-\eta) + F(\theta);$$

$$\therefore\ \frac{\Delta\omega + \Delta\theta}{c\sin(\theta + \omega)} = \sqrt{\frac{1 + \Delta\eta}{1 - \Delta\eta}}.$$

This form, though seldom occurring, is sometimes useful.

Hence also
$$\left.\begin{aligned}\sin\theta\,(\Delta\eta+\Delta\omega)&=(1+\Delta\theta)\sin(\eta-\omega),\\ \sin\theta.(\Delta\omega-\Delta\eta)&=(1-\Delta\theta)\sin(\eta+\omega)\end{aligned}\right\}.$$

3. To solve in the last Article, from the triplet of equations for η in terms of ω and θ, or for θ in terms of ω and η, we can proceed by easy elimination. To obtain $\sin\eta$, eliminate $\cos\eta$ between the two first equations of the triplet, or eliminate $\sin\eta$ and thus obtain $\cos\eta$. In the third equation of the triplet, substitute for $\cos\eta$ the value already obtained in terms of ω and θ, and you deduce $\Delta\eta$ in terms of ω and θ. All is merely algebraic routine. For conciseness put V^{-1} for $1-c^2\sin^2\omega\sin^2\theta$, then you have

$$\begin{cases}\sin\eta = V.\,\{\sin\omega\cos\theta\Delta\theta+\sin\theta\cos\theta\Delta\omega\}\\ \cos\eta = V.\,\{\cos\omega\cos\theta-\sin\omega\sin\theta\,.\,\Delta\omega\Delta\theta\}\\ \quad\Delta\eta = V.\,\{\Delta\omega\Delta\theta-c^2.\sin\omega\sin\theta\,.\cos\omega\cos\theta\}.\end{cases}$$

In these we change $\omega\eta$ to $-\eta$, $-\omega$ as before and thereby get $\sin\omega$, $\cos\omega$, $\Delta\omega$ in terms of η and θ.

4. Further, let $F\omega-F\theta=F\kappa$. Then in our results we may change $\theta\eta$ into $-\theta$, κ, which does not alter V; whence

$$\left.\begin{aligned}\sin\eta+\sin\kappa&=2V\sin\omega\cos\theta\Delta\theta\\ \cos\eta+\cos\kappa&=2V\cos\omega\cos\theta\end{aligned}\right\};$$

$$\therefore\quad \frac{\sin\eta+\sin\kappa}{\cos\eta+\cos\kappa}=\tan\omega\,.\,\Delta\theta.$$

But in Trigonometry the left hand

$$=\tan\tfrac{1}{2}(\eta+\kappa);$$

$$\therefore\quad \tan\tfrac{1}{2}(\eta+\kappa)=\tan\omega\,.\,\Delta\theta.$$

Exchange ω with θ, which changes κ to $-\kappa$. Thence

$$\tan\tfrac{1}{2}(\eta-\kappa)=\tan\theta\,.\,\Delta\omega.$$

If then ω and θ are given to find η, calculate $\frac{1}{2}(\eta+\kappa)$ and $\frac{1}{2}(\eta-\kappa)$ by the two last formulae and you obtain η from their sum.

Cor. 1. If $\omega=\theta$, or $2F\omega=F\eta$, you have

$$\kappa=0,\ \text{and}\ \tan(\tfrac{1}{2}\eta)=\tan\omega\,.\,\Delta\omega,$$

which gives η from ω. This is called Duplication.

Cor. 2. $\sin\eta-\sin\kappa=2V.\sin\theta\cos\omega\Delta\omega$, which is sometimes of avail.

5. *Bisection of the Nome $F(\eta)$.* For this we must solve

$$\tan(\tfrac{1}{2}\eta) = \tan\omega \,.\, \Delta\omega$$

for ω (Cor. 1 of last Art.) where η is given. Put $\tan(\tfrac{1}{2}\eta) = e$, *given* by hypothesis; and $\sin\omega = v$ unknown;

$$\therefore\ \tan\omega = \frac{v}{\sqrt{(1-v^2)}},$$

$$\Delta\omega = \sqrt{(1-c^2v^2)};$$

$$\therefore\ v\sqrt{\frac{1-c^2v^2}{1-v^2}} = e,$$

is to be solved for v; or

$$v^2(1-c^2v^2) = e^2(1-v^2),$$

$$c^2v^4 - e^2v^2 - v^2 + e^2 = 0,$$

$$c^2v^4 - (1+e^2)\,v^2 + \frac{(1+e^2)^2}{4c^2} = \frac{(1+e^2)^2}{4c^2} - e^2,$$

whence

$$\overline{1+e^2} - 2c^2v^2 = \sqrt{\{(1+e^2)^2 - 4c^2e^2\}} = (1+e^2)\sqrt{\left\{1 - c^2 \,.\, \frac{4e^2}{(1+e^2)^2}\right\}}.$$

But
$$\sin\eta = \frac{2e}{1+e^2};$$

$$\therefore\ (1+e^2) - 2c^2v^2 = (1+e^2)\sqrt{(1-c^2\sin^2\eta)} = (1+e^2)\Delta\eta,$$

and
$$2c^2v^2 = (1+e^2)(1-\Delta\eta).$$

Now
$$1+e^2 = 1+\tan^2.\tfrac{1}{2}\eta = \sec^2.\tfrac{1}{2}\eta = \frac{1}{\cos^2.\tfrac{1}{2}\eta} = \frac{2}{1+\cos\eta};$$

$$\therefore\ c^2v^2 = \frac{1-\Delta\eta}{1+\cos\eta}.$$

Again $(1-\Delta\eta)(1+\Delta\eta) = 1-\Delta^2\eta = c^2\sin\eta = c^2(1-\cos\eta)(1+\cos\eta)$;

$$\therefore\ \frac{1-\Delta\eta}{1+\cos\eta} = \frac{c^2(1-\cos\eta)}{1+\Delta\eta}.$$

Hence finally,
$$v^2 = \frac{1-\cos\eta}{1+\Delta\eta},$$

or if
$$c\sin\eta = \sin\zeta,\quad \Delta\eta = \cos\zeta;$$

$$v^2 = \frac{1-\cos\eta}{1+\cos\zeta} = \frac{2\sin^2.(\tfrac{1}{2}\eta)}{2.\cos^2.(\tfrac{1}{2}\zeta)},$$

whence
$$\sin\omega = \frac{\sin.(\tfrac{1}{2}\eta)}{\cos(\tfrac{1}{2}\zeta)}; \qquad \text{Q. E. I.}$$

$$\text{if}\quad \sin\zeta = c\sin\eta.$$

6.*** When $c \sin \eta = g$ is small, Legendre elegantly exhibits $\sin \omega$ and $\log \sin \omega$ in series of powers of g.

$$2 \cos \tfrac{1}{2}\zeta \text{ identically} = \sqrt{(1+\sin \zeta)} + \sqrt{(1-\sin \zeta)}.$$

Take the reciprocals of these,

$$\therefore \quad \tfrac{1}{2} \sec . \tfrac{1}{2}\zeta = \frac{\sqrt{(1+\sin \zeta)} - (1-\sin \zeta)}{2 \sin \zeta}.$$

But
$$\sin \zeta = c \sin \eta = g.$$

Then
$$\sec . \tfrac{1}{2}\zeta = \frac{\sqrt{(1+g)} - \sqrt{(1-g)}}{g}$$

$$= 2\left\{\tfrac{1}{2} + \frac{1.1.3}{2.4.6} g^2 + \frac{1.1.3.5.7}{2.4.6.8.10} g^4 + \&c.\right\},$$

whence $\sin \omega = \sin \tfrac{1}{2}\eta . \sec \tfrac{1}{2}\zeta = \sin(\tfrac{1}{2}\eta) . \{1 + \tfrac{1}{3}k_2 g^2 + \tfrac{1}{5}k_4 g^4 + \&c.\}$ (m).

Next, put $2z = \log\{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{(1-g^2)}\}$, and let z vary with g.

Observe that
$$1 - g^2 = \cos^2 \zeta,$$

$$\tfrac{1}{2} + \tfrac{1}{2}\sqrt{(1-g^2)} = \frac{1+\cos \zeta}{2} = (\cos \tfrac{1}{2}\zeta)^2 = \left(\frac{\sin \tfrac{1}{2}\eta}{\sin \omega}\right)^2.$$

Thus
$$z = \log(\sin \tfrac{1}{2}\eta) - \log \sin \omega,$$

and
$$\log \sin \omega = \log \sin . \tfrac{1}{2}\eta - z.$$

But
$$2dz = d \log(1 + \sqrt{1-g^2}) = \frac{d . \sqrt{(1-g^2)}}{1+\sqrt{(1-g^2)}};$$

in which
$$\frac{1}{1+\sqrt{(1-g^2)}} = \frac{1-\sqrt{(1-g^2)}}{g^2};$$

$$\therefore \quad 2dz = \frac{1-\sqrt{(1-g^2)}}{g^2} . \frac{-g dg}{\sqrt{(1-g^2)}} = -\{(1-g^2)^{-\frac{1}{2}} - 1\} . g^{-1} dg.$$

Expand by Binomial Theorem

$$2dz = -\{k_1 g + k_2 g^3 + k_3 g^5 + \ldots\} dg.$$

Integrate and you get

$$\log \sin \omega = \log \sin(\tfrac{1}{2}\eta) + \tfrac{1}{2}\left\{k_1 \frac{g^2}{2} + k_2 . \frac{g^4}{4} + k_3 . \frac{g^6}{6} + \&c.\right\} \quad \text{(n)},$$

where g means $c \sin \eta$, and the condition is

$$F(c\omega) = \tfrac{1}{2} F(c\eta).$$

COR. Equate the two values of $\log \sin \omega$. Deduct $\log \sin \frac{1}{2}\eta$ from both. Write x for g^2 which then is arbitrary, and you attain as an identity

$$\log\{1 + \tfrac{1}{3}k_2 x + \tfrac{1}{5}k_4 x^2 + \tfrac{1}{7}k_6 x^3 + \&c.\} = \tfrac{1}{4}k_1 x + \tfrac{1}{8}k_2 x^2 + \tfrac{1}{12}k_3 x^3 + \&c.$$

a notable equality, with x arbitrary.

7.$^{*}{}_{*}{}^{*}$ But other results of Arts. 3, 4 deserve notice. (1) We had

$$\tan\alpha = \tan\omega\Delta\theta;$$

if for conciseness $\qquad \alpha = \frac{1}{2}(\eta + \kappa)$.

Deduce $\cos\alpha$. We have

$$\sec^2\alpha = 1 + \tan^2\alpha = 1 + \tan^2\omega\Delta^2\theta = 1 + \tan^2\omega(1 - c^2\sin^2\theta)$$

$$= \sec^2\omega - c^2\tan^2\omega\sin^2\theta = \sec^2\omega(1 - c^2\sin^2\omega\sin^2\theta) = \sec^2\omega \,.\, V^{-1};$$

whence $\qquad \cos^2\alpha = \cos^2\omega \,.\, V$.

Exchange ω with θ, which alters κ to $-\kappa$, but does not affect V.

Thus from $\qquad \cos^2 . \frac{1}{2}(\eta + \kappa) = \cos^2\omega \,.\, V$,

we deduce $\qquad \cos^2 . \frac{1}{2}(\eta - \kappa) = \cos^2\theta \,.\, V$.

Eliminate V, and extract square root, then

$$\frac{\cos . \frac{1}{2}(\eta + \kappa)}{\cos . \frac{1}{2}(\eta - \kappa)} = \frac{\cos\omega}{\cos\theta} \quad \text{(a)}.$$

(2) Write $\sin\omega = p$, $\sin\theta = q$, $V^{-1} = 1 - c^2p^2q^2$.

We had $\qquad \sin\eta + \sin\kappa = 2V\sin\omega\cos\theta\Delta\theta$,

whence by exchanging ω and θ,

$$\sin\eta - \sin\kappa = 2V\sin\theta\cos\omega\Delta\omega,$$

V containing ω and θ symmetrically. Then

$$(\sin\eta + \sin\kappa)^2 - (\sin\eta - \sin\kappa)^2$$
$$= 4V^2\{\sin^2\omega\cos^2\theta\Delta^2\theta - \sin^2\theta\cos^2\omega\Delta^2\omega\}.$$

The quantity last in brackets

$$= p^2(1 - q^2)(1 - c^2q^2) - q^2(1 - p^2)(1 - c^2p^2)$$
$$= (p^2 - \overline{1 + c^2} \,.\, p^2q^2 + c^2p^2q^4) - (q^2 - \overline{1 + c^2} \,.\, q^2p^2 + c^2q^2p^4)$$
$$= (p^2 - q^2) + c^2p^2q^2(q^2 - p^2) = (p^2 - q^2)(1 - c^2p^2q^2) = (p^2 - q^2)V^{-1}.$$

Hence $\qquad 4\sin\eta \,.\, \sin\kappa = 4V^2 \,.\, (p^2 - q^2)V^{-1}$,

or $\qquad \sin\eta \,.\, \sin\kappa = V(\sin^2\omega - \sin^2\theta) \quad$ (b).

(3) Similarly, but with greater elaboration, we find

$$1 + \cos\eta \cos\kappa.$$

First $$\cos\eta + \cos\kappa = 2V\cos\omega\cos\theta,$$

and $$\cos\eta - \cos\kappa = -2V\sin\omega\sin\theta\Delta\omega\Delta\theta;$$

$$\therefore\ 4\cos\eta\cos\kappa = (\cos\eta + \cos\kappa)^2 - (\cos\eta - \cos\kappa)^2$$
$$= 4V^2\{(1-p^2)(1-q^2) - p^2q^2(1-c^2p^2)(1-c^2q^2)\}.$$

The quantity last in brackets

$$= (1 - p^2 - q^2 + p^2q^2) - p^2q^2(1 - c^2p^2 - c^2q^2 + c^4p^2q^2)$$
$$= (1 - c^4p^4q^4) - (p^2 + q^2 - c^2 . \overline{p^2+q^2} . p^2q^2)$$
$$= 1 - c^4p^4q^4 - (p^2+q^2)(1 - c^2p^2q^2),$$
$$\text{or}\quad 1 - c^4p^4q^4 - (p^2+q^2)V^{-1},$$

but $$1 - c^4p^4q^4 = (1 + c^2p^2q^2)(1 - c^2p^2q^2) = (1 + c^2p^2q^2)V^{-1}.$$

Hence $$\cos\eta\cos\kappa = V . (1 + c^2p^2q^2 - p^2 - q^2).$$

Add to this $$1 = V(1 - c^2p^2q^2);$$

$$\therefore\ 1 + \cos\eta\cos\kappa = V(2 - p^2 - q^2) = V(\cos^2\omega + \cos^2\theta) \quad \text{(c)}.$$

(4) Once more $$\Delta\eta + \Delta\kappa = 2V . \Delta\omega\Delta\theta,$$
$$\Delta\eta - \Delta\kappa = -2V . c^2\sin\omega\sin\theta\cos\omega\cos\theta;$$

$$\therefore\ 4\Delta\eta\Delta\kappa = (\Delta\eta + \Delta\kappa)^2 - (\Delta\eta - \Delta\kappa)^2$$
$$= 4V^2 . \{\Delta^2\omega . \Delta^2\theta - c^4\sin^2\omega\sin^2\theta\cos^2\omega\cos^2\theta\}.$$

The quantity in brackets

$$= (1 - c^2p^2)(1 - c^2q^2) - c^4p^2q^2(1 - p^2)(1 - q^2)$$
$$= \{1 - c^2 . \overline{p^2 + q^2} + c^4p^2q^2\} - c^4p^2q^2\{1 - (p^2+q^2) + p^2q^2\}$$
$$= (1 - c^4p^4q^4) - c^2 . \overline{p^2 + q^2} . (1 - c^2p^2q^2)$$
$$= V^{-1} . (1 + c^2p^2q^2) - c^2 . (p^2 + q^2) . V^{-1};$$

$$\therefore\ \Delta\eta . \Delta\kappa = V(1 + c^2p^2q^2 - c^2p^2 - c^2q^2).$$

Add $$1 = V(1 - c^2p^2q^2).$$

Hence $$1 + \Delta\eta . \Delta\kappa = V . (2 - c^2p^2 - c^2q^2) = V(\Delta^2\omega + \Delta^2\theta) \quad \text{(d)}.$$

8.*** In like manner it may be shown that

$$(1 \pm \sin\eta)(1 \pm \sin\kappa) = V(\cos\theta \pm \sin\omega\Delta\theta)^2,$$
$$(1 \pm \cos\eta)(1 \pm \cos\kappa) = V(\cos\omega \pm \cos\theta)^2,$$
$$\Delta\omega\Delta\theta = \Delta\eta + c^2 \sin\omega \sin\theta \cos\eta,$$
$$\Delta\eta \,.\, \Delta\omega\Delta\theta = b^2 + c^2 \cos\omega \cos\theta \cos\eta.$$

By far the most important among these is

$$\sin\eta \,.\, \sin\kappa = V \,.\, (\sin^2\omega - \sin^2\theta).$$

In the other notation (see Ch. III., Art. 1) if x, y are mesonomes of $\omega\theta$, and therefore $x + y$, $x - y$ mesonomes of $\eta\kappa$, this equation becomes

$$\sin|_c(x+y) . \sin|_c(x-y) = \frac{\sin^2|_c(x) - \sin^2|_c(y)}{1 - c^2 . \sin^2|_c(x) . \sin^2|_c(y)};$$

which is identically true, x *and* y *being independent.* Multiply it by $\sqrt{c} \,.\, \sqrt{c} = c$, and let $f\,(cx)$ mean $\sqrt{c}.\sin\,({}_c x)$; then this equation gives

$$f(c,\, x+y) \,.f(c,\, x-y) = \frac{[f\,(cx)]^2 - [f\,(cy)]^2}{1 - [f\,(cx) . f\,(cy)]^2};$$

x, y and c being all arbitrary.

9.*** PROBLEM. To *trisect* the complete Nome F_c. [For *Legendre's Scale,* which will be explained, this is an important Problem.] Write

$$F\,(c\alpha) = \tfrac{1}{3}F_c, \quad F\,(c,\,\beta) = \tfrac{2}{3}F_c.$$

Then α, β are Conjugate, since

$$F(\beta) + F(\alpha) = F_c.$$

Also $$F(\beta) - F(\alpha) = F(\alpha).$$

Hence for $\omega\theta$ of the preceding articles we may write $\beta\alpha$, and for $\eta\kappa$ we must put $\frac{1}{2}\pi$ and α. For

$$\tan\tfrac{1}{2}(\eta + \kappa) = \tan\omega\Delta\theta$$

we have $$\tan\tfrac{1}{2}(\tfrac{1}{2}\pi + \alpha) = \tan\beta \,.\, \Delta\alpha,$$

and for the law of Conjugation we get

$$b\tan\alpha \,.\, \tan\beta = 1.$$

From these two equations we are to deduce α and β in terms of c.

First eliminate $\tan\beta$. Then

$$b\tan\alpha \,.\, \tan\tfrac{1}{2}(\tfrac{1}{2}\pi + \alpha) = \Delta\alpha$$

or $$b\tan\alpha\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}} = \Delta\alpha.$$

The only unknown arc in the last is α. Let

$$k = \sin \alpha, \quad \tan \alpha = \frac{k}{\sqrt{(1-k^2)}};$$

$$\therefore \quad \frac{bk}{\sqrt{(1-k^2)}} \cdot \sqrt{\frac{1+k}{1-k}} = \sqrt{(1-c^2k^2)},$$

or

$$\frac{b^2k^2}{(1-k)^2} = 1 - c^2k^2.$$

If you put

$$b^2 = 1 - c^2,$$

and solve for c^2,

$$c^2 = \frac{2k-1}{(2-k)\,k^3}.$$

From this a TABLE *of trisection might be made by assuming values of k.* Observe, that if c^2, k be changed to c^{-2}, k^{-1}, the same equation is reproduced.—To solve $1 - 2k + c^2k^2(2k - k^2) = 0$ for k, begin by removing the term containing the 3rd power. Put $k = h + \frac{1}{2}$,

$$\therefore \quad h^4 - \tfrac{3}{2}h^2 + (2c^{-2} - 1)\,h - \tfrac{3}{16} = 0.$$

Identify this with $\quad (h^2 - ph + q)(h^2 + ph - r)$;

$$\therefore \quad q - r = p^2 - \tfrac{3}{2};$$

$$qr = \tfrac{3}{16};$$

$$p(q+r) = 2c^{-1} - 1.$$

Eliminate q, r, and you get exactly

$$(p^2 - 1)^3 = \frac{4b^2}{c^4} \text{ [the reciprocal of } m^2 \text{ in p. 20 above].}$$

Put

$$n = \sqrt[3]{\frac{4b^2}{c^4}};$$

$$\therefore \quad p^2 = 1 + n,$$

$$q - r = n - \tfrac{1}{2};$$

$$q + r = \sqrt{\frac{1-n}{1+n}}.$$

For k we want the smallest root, and that positive. We get it by solving $\quad h^2 + ph = r.$

Cor. 1. When c is infinitesimal, $2k - 1 = 0$, $\alpha = \frac{1}{6}\pi = 30°$, the *lower* limit.

Cor. 2. When $c = 1$, $b = 0$, $n = 0$, $p = 1$, $-q = r = \frac{1}{4}$, $h^2 + h = \frac{1}{4}$, $h + \frac{1}{2}$ or $k = \sqrt{\frac{1}{2}} = \sin 45°$. This is the *upper* limit, to which α tends as c increases.

10. Since $F(\beta) = 2F(\alpha)$, $\tan \frac{1}{2}\beta = \tan \alpha \, . \, \Delta\alpha$.

We had $$\tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\alpha) = \tan \beta \, . \, \Delta\alpha.$$

To eliminate $\Delta\alpha$ from these, eliminates c entirely. The result is unexpectedly simple. First

$$\tan \alpha \, . \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\alpha) = \tan \beta \, . \tan \tfrac{1}{2}\beta.$$

We have $k = \sin \alpha$; for a moment put $\lambda = \cos \beta$, then

$$\frac{k}{\sqrt{(1-k^2)}} \cdot \sqrt{\frac{1+k}{1-k}} \text{ (by Trigonometry)} = \frac{\sqrt{(1-\lambda^2)}}{\lambda} \cdot \sqrt{\frac{1-\lambda}{1+\lambda}}.$$

Square; $$\therefore \frac{k^2}{(1-k)^2} = \frac{(1-\lambda)^2}{\lambda^2};$$

or $$k\lambda = (1-k)(1-\lambda),$$

whence $$k + \lambda = 1,$$

or $$\sin \alpha + \cos \beta = 1.$$

This linear equation, with the conjugate $\cot \alpha . \cot \beta = b$, contains everything. Of course we have

$$c^2 = \frac{2k-1}{(2-k)k^3}$$

with $$\sin \alpha = k, \text{ and } \cos \beta = 1 - k,$$

so that $$2k - 1 = \sin \alpha - \cos \beta, \ \cos^2 \beta = 1 - 2k + k^2,$$

$$\therefore \ 2k - k^2 = 1 - \cos^2 \beta = \sin^2 \beta.$$

Hence $$c^2 = \frac{\sin \alpha - \cos \beta}{\sin^2 \alpha \, . \, \sin^2 \beta}.$$

A Table of Trisection seemed awhile the desideratum.

• 11. *Addition of Epinomes.* PROBLEM. To find the relation of $E\omega$ to $E\theta$ when $F\omega + F\theta = F\eta$. If we differentiate

$$cF(\omega) + cF(\theta) = cF(\eta)$$

with ω and θ constant, but c variable (in which case η must vary with c), we get from Ch. I. Art. 11,

$$b^2 . \frac{d(c.F\omega)}{dc} + b^2 . \frac{d(c.F\theta)}{dc} = b^2 . \frac{d(c.F\eta)}{dc} + b^2 . \frac{d(c.F\eta)}{d\eta} . \frac{d\eta}{dc};$$

where $\frac{d\eta}{dc}$ must be estimated from Art. 3 above, p. 54. Then its

equivalent is

$$(E\omega - c^2 \sin\omega \sin\omega^0) + (E\theta - c^2 \sin\theta \sin\theta^0)$$
$$= (E\eta - c^2 \sin\eta \sin\eta^0) + b^2 \cdot \frac{c}{\Delta\eta} \cdot \frac{d\eta}{dc}.$$

Evidently then, if we put $P = E\omega + E\theta - E\eta$,

P will be trigonometrical. We find its value most easily by supposing that c and η are constant, making a new beginning.

Then

$$dP = dE\omega + dE\theta = \Delta\omega d\omega + \Delta\theta d\theta = \frac{\cos\omega - \cos\theta\cos\eta}{\sin\theta \cdot \sin\eta} \cdot d\omega$$
$$+ \frac{\cos\theta - \cos\omega\cos\eta}{\sin\omega\sin\eta} d\theta,$$

whence $\sin\omega \sin\theta \sin\eta \cdot dP = (\sin\omega\cos\omega d\omega + \sin\theta\cos\theta d\theta)$

$$- \cos\eta\,(\sin\omega\cos\theta d\omega - \sin\theta\cos\omega d\omega)$$
$$= \tfrac{1}{2} d\,(\sin^2\omega + \sin^2\theta + \sin^2\eta) + d\,(\cos\eta\cos\omega\cos\theta);$$

where $\sin^2\eta$ is a constant which we may add.

But formula (A) in p. 53, Art. 2, reduces the last to

$$\tfrac{1}{2}c^2 \cdot d\,(\sin^2\omega \sin^2\theta \sin^2\eta),$$

that is to $\qquad c^2 \cdot \sin\omega\sin\theta\sin\eta \cdot d\,(\sin\omega\sin\theta\sin\eta)$.

Divide by $\sin\omega\sin\theta\sin\eta$ on each side, then

$$dP = c^2 \cdot (\sin\omega\sin\theta\sin\eta).$$

Integrate, then $\qquad P = c^2 \sin\omega\sin\theta\sin\eta + c_1$.

But when $\qquad \theta = 0, \quad \omega = \eta,$

$$\therefore\ P = 0, \quad \therefore\ c_1 = 0,$$

i.e. no *constant of integration* is needed.

Thus we find $\quad E\omega + E\theta - E\eta = c^2 \sin\omega\sin\theta\sin\eta$.

COR. Since also $\qquad F\omega + F\theta - F\eta = 0,$

every function of ϕ made up of $E - aF$, where a is any constant, makes also $\qquad \phi\omega + \phi\theta - \phi\eta = c^2 \sin\omega\sin\theta\sin\eta$.

In particular, when $a = \aleph_c$ we have

$$G\omega + G\theta - G\eta = c^2 \sin\omega\sin\theta\sin\eta.$$

12. We cannot similarly obtain $\urcorner\omega + \urcorner\theta - \urcorner\eta$. If we attempt it, we only alight upon Π: nevertheless when $\omega = \theta$, we can find $2\urcorner\omega - \urcorner\eta$. This, from $2F\omega = F\eta$, gives

$$\sin\eta = \frac{2\sin\omega\cos\omega\Delta\omega}{1 - c^2\sin^4\omega}$$

from Art. 3. Also $2G\omega - G\eta = c^2\sin^2\omega\sin\eta$.

Multiply by $$2dF\omega = dF\eta = \frac{2d\omega}{\Delta\omega}.$$

Then $$4G\omega \,.\, dF\omega - G\eta \,.\, dF\eta = c^2\sin^2\omega \,.\, \sin\eta\,\frac{2d\omega}{\Delta\omega}.$$

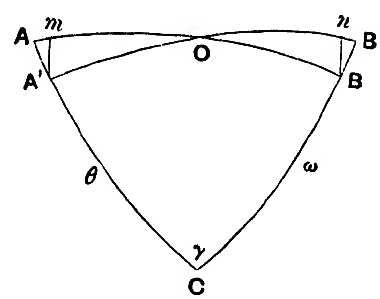
Give to $\sin\eta$ its equivalent in the last term,

$$\therefore\ 4d\urcorner\omega - d\urcorner\eta = \frac{4c^2\sin^3\omega\cos\omega d\omega}{1 - c^2\sin^4\omega},$$

whence $$4\urcorner\omega - \urcorner\eta = -\log(1 - c^2\sin^4\omega) \qquad \text{Q. E. I.}$$

when $2F\omega = F\eta$.

13. Legendre also deduces Euler's main equation of Art. 2 by Spherical Trigonometry very simply. Let $\omega\theta\eta$ be the sides and $\alpha\beta\gamma$ the opposite angles of a spherical triangle ABC. Put



$$c = \frac{\sin\alpha}{\sin\omega} = \frac{\sin\beta}{\sin\theta} = \frac{\sin\gamma}{\sin\eta};$$

and let c be constant. If also η be constant, so is γ. When $\omega\theta\alpha\beta$ vary, ω decreases when θ increases, and if AB change to $A'B'$, the two lines cross, as in O. Drop Bn perpendicular to $A'OB'$, and $A'm$ perpendicular to AOB. Ultimately $OA' = Om$, the angle at O being infinitesimal, so $OB = On$,

$$\therefore\ mB = A'n.$$

But $$AB = \eta = A'B'$$

since η is constant; $\therefore\ Am = nB'$.

But $$A'A = -\delta\theta, \quad BB' = \delta\omega;$$

$$\therefore\ Am = -\delta\theta \,.\, \cos\alpha, \quad nB' = \delta\omega \,.\, \cos\beta.$$

Hence the equation $$Am = nB'$$

gives $$\delta\omega \cos\beta + \delta\theta \cos\alpha = 0,$$

or $$\frac{\delta\omega}{\cos\alpha} + \frac{d\theta}{\cos\beta} = 0.$$

Again, since
$$\sin\alpha = c\sin\omega,$$
$$\sin\beta = c\sin\theta,$$
$$\sin\gamma = c\sin\eta,$$

we have (with only signs doubtful)
$$\pm\cos\alpha = \Delta\omega,$$
$$\pm\cos\beta = \Delta\theta,$$
$$\pm\cos\gamma = \Delta\eta.$$

Also we know $\delta\omega$, $\delta\theta$ to have opposite signs, therefore the true equation is
$$\frac{d\omega}{\Delta\omega} + \frac{d\theta}{\Delta\theta} = 0,$$

whence $$F(c\omega) + F(c\theta) = const.$$

But when $\theta = 0$ BA coincide with BC and $\omega = \eta$;
$$\therefore \quad F(c\omega) + F(c\theta) = F(c, \eta),$$

under the condition that
$$c = \frac{\sin\gamma}{\sin\eta} \text{ or } \pm\cos\gamma = \Delta(c\eta).$$

Also by the known property of Spherics,
$$\cos\eta = \cos\omega\cos\theta + \sin\omega\sin\theta\cos\gamma.$$

Only the sign of $\cos\gamma$ remains doubtful. To decide this, suppose γ to be almost 180°, $\therefore$ η is almost $\omega + \theta$. This shows the negative sign to be right, or $\cos\gamma = -\Delta(c\eta)$.

Hence we have $\cos\eta = \cos\omega\cos\theta - \sin\omega\sin\theta \,.\, \Delta(c, \eta)$

with $$F\omega + F\theta = F\eta.$$

CHAPTER V.

SCALE OF INDEX 2.

1. *Lagrange's Problem.* Given the equation

$$F(c\omega) = \mu \, . \, F(h, \theta),$$

where c, h, μ are constant, and ω, θ variable circular arcs; find an algebraic relation between the constants; also between the five elements.—The following investigation, when I began these studies, pleased me as very elementary and unartificial. As such, it may please other beginners.

We know that if $t = \tan \omega$, and $u = \tan \theta$, with

$$c^2 + b^2 = 1, \text{ and } h^2 + g^2 = 1,$$

the given equation yields $\dfrac{dt}{f(b, t)} = \dfrac{\mu du}{f(gu)}$,

if $f.(b, t)$ means $\qquad \surd(1 + t^2) \surd(1 + b^2t^2)$.

Also by Binomial Theorem the reciprocal of the last has the *form*

$$1 - A_1t^2 + A_2t^4 - \&c.$$

So too the reciprocal of $f(g, u)$ has the *form*

$$1 - B_1u^2 + B_2u^4 - B_3u^6 + \&c.$$

Introduce these series, and integrate then

$$(t - \tfrac{1}{3}A_1t^3 + \tfrac{1}{5}A_2t^4 - \&c.) = \mu(u - \tfrac{1}{3}B_1u^3 + \tfrac{1}{5}u^5 - \&c.).$$

In the *given* equation ω and θ vanish together, and in vanishing $\mu u = t$.

When t^4 and u^4 are negligible,

$$\mu u = t \, . \, \frac{1 - \frac{1}{2}A_1t^2}{1 - \frac{1}{3}B_1u^2}.$$

On the right, insert $u^2 = \mu^{-2}t^2$ as a first approximation, then

$$\mu u = t \cdot \frac{3 - A_1 t^2}{3 - B_1 \cdot \mu^{-2} t^2}.$$

This suggests as possible solutions, to try the form

$$\mu u = t \cdot \frac{1 - mt^2}{1 - rt^2},$$

or simpler still,
$$\mu u = \frac{t}{1 - rt^2};$$

not but that
$$\mu u = t \cdot \frac{\psi(t)}{\phi(t)}$$

seems a possible form, with ψ and ϕ rational functions.

2. Start then tentatively with

$$u = \frac{\mu^{-1} t}{1 - rt^2};$$

and seek relations of these elements by actual trial, r of course is constant. This assumption yields

$$1 + u^2 = \frac{1 + (\mu^{-2} - 2r)\, t^2 + r^2 t^4}{(1 - rt^2)^2};$$

and to simplify, assume $\mu^{-2} - 2r = 1 + r^2$, which reduces the numerator to $(1 + t^2)(1 + r^2 t^2)$, and our assumption is exactly $\mu^{-1} = 1 + r$.

Next, we deduce

$$1 + g^2 u^2 = \frac{1 + (\mu^{-2} g^2 - 2r)\, t^2 + r^2 t^4}{(1 - rt^2)^2}.$$

Again, to simplify, assume $\mu^{-2} g^2 - 2r = 2r$, which reduces the numerator to $(1 + rt^2)^2$; and our new assumption means

$$g^2 = 4r \cdot \mu^2 = \frac{4r}{(1 + r)^2};$$

whence
$$h^2 = \left(\frac{1 - r}{1 + r}\right)^2.$$

Now also we have two results,

$$\sqrt{(1 + u^2)} = \frac{\sqrt{(1 + t^2)}\, \sqrt{(1 + r^2 t^2)}}{1 - rt^2} \quad \text{(L)},$$

and
$$\sqrt{(1 + g^2 u^2)} = \frac{1 + rt^2}{1 - rt^2} \quad \text{(M)}.$$

Thirdly we must seek μdu in terms of t; from our assumption

$$u = \frac{\mu^{-1} t}{1 - rt^2};$$

$$\therefore \quad \mu du = d \cdot \frac{t}{1 - rt^2} = \text{by routine } \frac{(1 + rt^2)\, dt}{(1 - rt^2)^2} \quad \text{(N)}.$$

Now $$\mu \cdot dF(h, \theta) = \frac{\mu du}{\sqrt{(1 + u^2)}\sqrt{(1 + g^2 u^2)}}.$$

On the right of this last, insert the equivalents from (L), (M), (N),

and the result is $$\frac{dt}{\sqrt{(1 + t^2)}\sqrt{(1 + r^2 t^2)}},$$

so that if we make a *third* assumption $r = b$ we obtain

$$\mu d \cdot F(h, \theta) = dF(c\omega).$$

And with disposable constants $c\, h\, \mu\, r$ we have a right to make three assumptions. Then, integrating, we find exactly

$$\mu F(h, \theta) = F(c\omega),$$

which solves our problem.

3. Many and important inferences follow. First, observe that

$$h = \frac{1 - b}{1 + b},$$

so that the *constants* are related as in Landen's scale, and

$$\mu^{-1} = 1 + b, \quad \mu = \tfrac{1}{2}(1 + h).$$

I. Thus $$F(c\omega) = \tfrac{1}{2}(1 + h)\, F(h, \theta),$$

and $$F(h, \theta) = (1 + b)\, F(c\omega).$$

II. Make ω increase in $u = \dfrac{(1 + b)\, t}{1 - bt^2}$. In fact, so long as $1 - bt^2$ is positive, so is u. When $t = \sqrt{b^{-1}}$, u becomes infinite, therefore $\theta = \frac{1}{2}\pi$. When t exceeds $\sqrt{b^{-1}}$, u becomes negative. When $\omega = \frac{1}{2}\pi$, t is infinite and u (previously negative) sinks to zero, $\therefore$ θ becomes π, at the crisis of $\omega = \frac{1}{2}\pi$. [When $t^2 = \sqrt{b}$, $\omega =$ conjugate ω^0.]

III. Introduce these values into I., and you have

$$F_c = \tfrac{1}{2}(1 + h)\, F(h, \pi) \text{ or } (1 + h)\, F_h,$$

a relation already found, in Chap. II.

IV. Write $\tan\psi$ for $b\tan\omega$; then

$$\tan\theta = \frac{(1+b)\,t}{1-bt^2} = \frac{\tan\psi + \tan\omega}{1-\tan\psi\tan\omega} = \tan(\psi+\omega),$$

whence simply $\qquad \theta = \psi + \omega$, or $\theta - \omega = \psi$,

that is $\qquad \tan(\theta-\omega) = \tan\psi = b\tan\omega.$

This neatly deduces θ when ω is given, and shows, when b constantly lessens, that $\tan(\theta-\omega)$ converges to zero, and $(\theta-\omega)$ to 0, else to $n\pi$.

V. Moreover, if ω increase perpetually from 0 to $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$, &c. the t accompanying becomes *zero* and *infinite* alternately at the end of each quadrant; so that, each time ω receives $\frac{1}{2}\pi$ for increment, θ has π.

On this account the number 2 is called the *Index* of this Scale.

4. From ω given, we now deduce functions of θ by routine.

I. From $\tan\theta$, we get

$$1 + u^2 = 1 + \left(\frac{\overline{1+b}\,.\,t}{1-bt^2}\right)^2.$$

Now $(1-bt^2)^2 + (\overline{1+b}\,.\,t)^2 = 1 + t^2 + b^2t^2 + b^2t^4 = (1+t^2)(1+b^2t^2)$,

so that $\qquad \sec\theta = \dfrac{\sqrt{(1+t^2)}\sqrt{(1+b^2t^2)}}{1-bt^2}.$

Reciprocating $\qquad \cos\theta = \cos\omega\,.\,\dfrac{1-bt^2}{\sqrt{(1+b^2t^2)}};$

whence, multiplying numerator by $\cos\omega$ and denominator by

$$\cos\omega = \sqrt{\cos^2\omega},$$

II. $$\cos\theta = \frac{\cos^2\omega - b\sin^2\omega}{\sqrt{(\cos^2\omega + b^2\sin^2\omega)}} = \frac{\cos^2\omega - b\sin^2\omega}{\Delta(c\omega)}.$$

Also $\qquad \sin\theta = \dfrac{\tan\theta}{\sec\theta}$ and $\tan\theta = \dfrac{(1+b)\,t}{1-bt^2};$

III. $$\therefore\ \sin\theta = \frac{(1+b)\,t}{\sqrt{(1+t^2)}\sqrt{(1+b^2t^2)}} = \frac{(1+b)\sin\omega}{\sqrt{(1+b^2t^2)}}$$

$$= \frac{(1+b)\sin\omega}{\sqrt{(\sec^2\omega - c^2\tan^2\omega)}} = (1+b)\sin\omega\,.\,\frac{\cos\omega}{\Delta(c\omega)}.$$

More elegantly, if ω^0 is Conjugate to ω, $\sin\theta = (1+b)\sin\omega\,.\,\sin\omega^0$.

Or again, since $\qquad 1+b = \dfrac{2}{1+h} = \dfrac{c}{\sqrt{h}},$

$$\therefore\ \sqrt{h}\sin\theta = \sqrt{c}\,.\,\sin\omega\,.\,\sqrt{c}\sin\omega^0.$$

IV. Further, we had (in Art. 2) $\sqrt{1+g^2u^2}=\dfrac{1+bt^2}{1-bt^2}$.

Introduce $$u=\frac{\sin\theta}{\cos\theta},\quad t=\frac{\sin\omega}{\cos\omega},\text{ herein;}$$

then $$\frac{\sqrt{(\cos^2\theta+g^2\sin^2\theta)}}{\cos\theta},$$

that is, $$\frac{\Delta(h,\ \theta)}{\cos\theta}=\frac{\cos^2\omega+b\sin^2\omega}{\cos^2\omega-b\sin^2\omega},\text{ also}=\frac{1+bt^2}{1-bt^2}.$$

Eliminate $\cos\theta$; then $\Delta(h,\ \theta)=\dfrac{\cos^2\omega+b\sin^2\omega}{\Delta(c,\ \omega)}$.

5. V. By law of Conjugates, $\cot\omega^0=b\tan\omega=\tan\psi$ of Art. 3,

$$\therefore\quad \omega^0=\tfrac{1}{2}\pi-\psi.$$

We had $\theta=\omega+\psi$, hence $\theta=\omega+(\frac{1}{2}\pi-\omega^0)$.

VI. $$\cot\theta=\frac{1}{\tan\theta}=\frac{1-b\tan^2\omega}{(1+b)\tan\omega}=\frac{\cot\omega-b\tan\omega}{1+b}$$
$$=\frac{1+h}{2}\cot\omega-\frac{1-h}{2}.\tan\omega$$
$$=\tfrac{1}{2}(\cot\omega-\tan\omega)+\tfrac{1}{2}h(\cot\omega+\tan\omega)=\cot2\omega+h\operatorname{cosec}2\omega.$$

VII. We may now add
$$\frac{\Delta(h\theta)}{\Delta(c\omega)}=\frac{\cos^2\omega+b\sin^2\omega}{\Delta^2(c\omega)}=\frac{1+b\tan^2\omega}{1+b^2\tan^2\omega}.$$

The equation $\tan\psi=b\tan\omega$ in Chap. III. Art. 5 was developed for ψ in series. Here we have only to replace ψ by $\theta-\omega$, and we obtain $$\theta-\omega=\omega-h\sin2\omega+\tfrac{1}{2}h^2\sin4\omega-\tfrac{1}{3}h^3\sin6\omega+\&c.$$
Besides, we have
$$\cos\theta=\sin(\omega^0-\omega),\quad \sin\theta=\cos(\omega^0-\omega).$$
Further
$$(1+b)\Delta(h\theta)=(1+b).\frac{1-(1-b)\sin^2\omega}{\Delta(c\omega)}$$
$$=\frac{(1+b)-c^2\sin^2\omega}{\Delta(c\omega)}=\frac{(1-c^2\sin^2\omega)+b}{\Delta(c\omega)}=\Delta(c\omega)+\frac{b}{\Delta(c\omega)},$$
that is, since $$\frac{b}{\Delta(c\omega)}=\Delta(c\omega^0),$$
$$(1+b)\Delta(h\theta)=\Delta(c\omega)+\Delta(c\omega^0).$$

Similarly

$$(1-b)\cos\theta = (1-b)\,.\,\frac{1-(1+b)\sin^2\omega}{\Delta(c\omega)}$$

$$= \frac{(1-b)-c^2\sin^2\omega}{\Delta(c\omega)} = \Delta(c\omega) - \Delta(c\omega^0).$$

6. Next, suppose θ given to find ω. Identically, we have

$$\sin(\omega \pm \psi) = \sin\omega\cos\psi \pm \cos\omega\sin\psi = (1 \pm \cot\omega\tan\psi)\sin\omega\cos\psi.$$

But by hypothesis $\qquad \tan\psi = b\tan\omega,$

$$\therefore \quad \cot\omega\tan\psi = b;$$

or $\qquad \sin(\omega \pm \psi) = (1 \pm b)\sin\omega\cos\psi,$

whence $\qquad \dfrac{\sin(\omega-\psi)}{\sin(\omega+\psi)} = \dfrac{1-b}{1+b} = h.$

But $\qquad \omega + \psi = \theta,$

$$\therefore \quad \omega - \psi = 2\omega - \theta.$$

Finally $\qquad \sin(2\omega - \theta) = h\sin\theta.$

This is the simplest equation for finding ω from θ.

When h is evanescent $2\omega - \theta = 0$, as before shown by the series for θ.

It is sometimes convenient to write Δ, Δ_1 for $\Delta(c\omega)$, $\Delta(h\theta)$ and Δ^0 for $\Delta(c\omega^0)$.

Eliminate Δ^0 from $(1+b)\Delta_1 = \Delta + \Delta^0$ and $(1-b)\cos\theta = \Delta - \Delta^0$,

$$\therefore \quad (1+b)\Delta_1 + (1-b)\cos\theta = 2\Delta,$$

or $\qquad \dfrac{2}{1+b}\Delta = \Delta_1 + \dfrac{1-b}{1+b}\cos\theta;$

$$\text{i.e.} \quad (1+h)\Delta = \Delta_1 + h\cos\theta.$$

Multiply the last by H (modular to h), observing that $(1+h)H = C$.

Then $\qquad C\Delta = H\Delta_1 + hH\cos\theta \;\text{ or }\; C_1\Delta_1 + c_1C_1\cos\theta$

in our notation for Landen's scale. But to complete the analogy we ought to denote θ by ω_1 when $F(c\omega) = \mu F(c_1\omega_1)$ and $\mu^{-1} = 1 + b$, or $\mu = \frac{1}{2}(1 + c_1)$.

We had also by Art. 3,

$$\frac{\Delta_1}{\cos\theta} = \frac{1+bt^2}{1-bt^2},$$

$$\therefore \quad bt^2 = \frac{\Delta_1 - \cos\theta}{\Delta_1 + \cos\theta} = \frac{(\Delta_1 - \cos\theta)^2}{\Delta_1^{\,2} - \cos^2\theta}.$$

The new denominator is

$$(1-h^2\sin^2\theta)-\cos^2\theta=\sin^2\theta-h^2\sin^2\theta=g^2\sin^2\theta;$$

hence $$\sqrt{b}\,.\tan\omega=\frac{\Delta_1-\cos\theta}{g\sin\theta}\ldots\ldots\ldots\ldots\ldots\ldots(a).$$

Again, since $(\Delta_1+\cos\theta)(\Delta_1-\cos\theta)=g^2\sin^2\theta$,

the last equally gives $\sqrt{b}\tan\omega=\dfrac{g\sin\theta}{\Delta_1+\cos\theta}$.

Further if $$F(h\theta^0)+F(h\theta)=F_h,$$

$$\sin\theta^0=\frac{\cos\theta}{\Delta_1}=\frac{1-bt^2}{1+bt^2}.$$

Thence in reverse, $$bt^2=\frac{1-\sin\theta^0}{1+\sin\theta^0},$$

or $$\left.\begin{aligned}\sqrt{b}\tan\omega&=\tan(\tfrac14\pi-\tfrac12\theta^0),\\ \text{whence also}\quad\sqrt{b}\tan\omega^0&=\tan(\tfrac14\pi+\tfrac12\theta^0).\end{aligned}\right\}$$

Observe also that $\cos\theta^0=\dfrac{g\sin\theta}{\Delta_1}$, by law of Conjugates,

$$\therefore\ \sec\theta^0=\frac{\Delta_1}{g\sin\theta}\ldots\ldots\ldots\ldots\ldots\ldots(b).$$

Also $$\tan\theta^0\equiv\frac{1}{g\tan\theta}=\frac{\cos\theta}{g\sin\theta}\ldots\ldots\ldots\ldots\ldots\ldots(c),$$

so that by (*a*), (*b*), (*c*) $\sqrt{b}\tan\omega$ or $\dfrac{\Delta_1-\cos\theta}{g\sin\theta}$

is expressible as
$$\left.\begin{aligned}\sqrt{b}\tan\omega&=\sec\theta^0-\tan\theta^0,\\ \text{yielding}\quad\sqrt{b}\tan\omega^0&=\sec\theta^0+\tan\theta^0.\end{aligned}\right\}$$

But all this is mere preparation.

We proceed to calculate F, E, ꓶ from ω and c given.

Calculations by Lagrange's Scale.

7. *Calculation of the Nome.* We have $F(c\omega)=Cx$, and we presume the Modular C to be known. We need only to calculate x the Mesonome. Suppose amplitudes $\omega\,\omega_1\,\omega_2\,\omega_3\ldots$ to be computed successively by the same law,

$$\tan(\omega_1-\omega)=b\tan\omega,\quad\tan(\omega_2-\omega_1)=b_1\tan\omega_1,$$

and so on. We know that the series $b\,b_1\,b_2\,b_3\ldots$ rapidly runs towards 1.

We found $$F = \tfrac{1}{2}(1+h)F_1(\omega) \text{ indefinite}$$
and $$F_c = (1+h)F_h,$$
or $$C = (1+h)H;$$
$$\therefore \frac{F}{C} = \tfrac{1}{2}\cdot\frac{F_1}{H} \text{ or } \tfrac{1}{2}\cdot\frac{F_1}{C_1}.$$
Now $$x = \frac{F}{C}.$$
Similarly we must make
$$x_1 = \frac{F_1}{C_1}, \quad x_2 = \frac{F_2}{C_2},$$
as successive Mesonomes. *Then we deduce*
$$x = \tfrac{1}{2}x_1, \quad x_1 = \tfrac{1}{2}x_2, \quad x_2 = \tfrac{1}{2}x_3, \text{ \&c.}$$
or $$x = 2^{-1}x_1 = 2^{-2}x_2 = \ldots = 2^{-n}x_n.$$
Also when n is so large that c_n is insensible, $F_n = \omega_n$ and $x_n = F_n$. Thus with a large n we have sensibly $x_n = \omega_n$; and x (which $= 2^{-n}x_n$ *accurately*) is $2^{-n}\omega_n$ *approximately*. Thus x, which alone we need to find, is the limit of ω, $2^{-1}\omega_1$, $2^{-2}\omega_2$, $2^{-3}\omega_3$ ..., and all is reduced to the calculation of ω_1 ω_2 ω_3 ... In determining their values from the trigonometrical equations it must be remembered that each *tends* to be double of the preceding, which will guide us to know how often 2π is to be added to the arc whose *tangent* is known.

8. *Calculation of the Nomiscus*, $G(c\omega)$, from which we obtain
$$E = \aleph_c . F + G.$$
We must first apply Lagrange's scale to it. Since by hypothesis and μ known,
$$\frac{d\omega}{\Delta} = \frac{1+h}{2}\cdot\frac{d\theta}{\Delta_1},$$
multiply by $(1+h)^2 . \Delta^2 = (\Delta_1 + h\cos\theta)^2$ of Art. 6. Then
$$(1+h)^2 . \Delta d\omega = \frac{1+h}{2} . (\Delta_1^2 + 2h\Delta_1\cos\theta + h^2\cos^2\theta) . \frac{d\theta}{\Delta_1}.$$
But $$h^2\cos^2\theta = h^2 - h^2\sin^2\theta = \Delta_1^2 - g^2,$$
so that $$(1+h)\Delta d\omega = (2\Delta_1^2 - g^2 + 2h\Delta_1\cos\theta) . \frac{d\theta}{2\Delta_1}$$
$$= \Delta_1 d\theta - \tfrac{1}{2}g^2 . \frac{d\theta}{\Delta_1} + h\cos\theta d\theta.$$
Integrate; then $$(1+h)E \doteq E_1 - \tfrac{1}{2}g^2F_1 + h\sin\theta.$$

In this make $\omega = \tfrac{1}{2}\pi$, $\theta = \pi$;

$$\therefore \quad (1+h)\,E_c = 2E_h - \tfrac{1}{2}g^2 \,.\, 2F_h.$$

Multiply the last by $\dfrac{F}{F_c} = \tfrac{1}{2} \,.\, \dfrac{F_1}{F_h}$,

and subtract from the preceding; whence $(1+h)\,G = G_1 + h\sin\theta$.

Multiply the last by H, observing that $(1+h)\,H = C$,

$$\therefore \quad CG = HG + hH\sin\theta,$$

or in continuous notation $CG = C_1G_1 + c_1C_1\sin\omega_1$.

This is the equation of reduction.

Add together $(CG - C_1G_1)$, $(C_1G_1 - C_2G_2)$, $(C_2G_2 - C_3G_3)\ldots$ up to $(C_{n-1}G_{n-1} - C_nG_n)$ and their equivalents. Then

$$CG - C_nG_n = c_1C_1\sin\omega_1 + c_2C_2\sin\omega_2 + \ldots + c_nC_n\sin\omega_n.$$

When c_n is insignificant, so is G_n, and $C_n = 1$. Finally then

$$CG = c_1C_1\sin\omega_1 + c_2C_2\sin\omega_2 + c_3C_3\sin\omega_3 + \&c.\ \textit{ad infin.}$$
$$= \Sigma\,(c_nC_n\sin\omega_n), \text{ with } n = 1, 2, 3, 4\ldots$$

which is very easy to remember.

But to apply this formula, we may observe that, from $(1+c_1)\,C_1 = C$

and $c = \dfrac{2\sqrt{c_1}}{1+c_1}$, $\qquad \dfrac{c_1C_1}{cC} = \dfrac{c_1}{c\,(1+c_1)} = \tfrac{1}{2}\sqrt{c_1}$ .... (a),

so that $G = c\,\{\tfrac{1}{2}\sqrt{c_1}\sin\omega_1 + \tfrac{1}{4}\sqrt{c_1c_2}\sin\omega_2 + \tfrac{1}{8}\sqrt{c_1c_2c_3}\sin\omega_3$ &c.

Suppose $\omega_n{}^0$ conjugate to ω_n with modulus c_n. Then since generally

$$\sqrt{c_1}\sin\omega_1 = \sqrt{c}\sin\omega \,.\, \sqrt{c}\sin\omega^0,$$

$$\frac{n^{\text{th}}\text{ term of series}}{(n-1)^{\text{th}}\text{ term}} = \tfrac{1}{2}\,\frac{\sqrt{c_n}\sin\omega_n}{\sin\omega_{n-1}} = \tfrac{1}{2}\,c_{n-1}\sin\omega^0{}_{n-1}.$$

This fraction measures the convergence; and it is less than $\tfrac{1}{2}c_{n-1}$.

9. *Calculation of the Diplonome* Observe that, by (a) above,

$$c_1C_1\sin\omega_1 = \tfrac{1}{2}cC\sqrt{c_1}\sin\omega_1 = \frac{c^2C\sin\omega\cos\omega}{2\Delta\,(c\omega)}.$$

Hence $$CG = C_1G_1 + \tfrac{1}{2}C\,.\,\frac{c^2\sin\omega\cos\omega}{\Delta\,(c\omega)}.$$

Multiply by $$\frac{dF}{C} = \tfrac{1}{2}\,\frac{dF_1}{C_1} = \frac{d\omega}{C\,.\,\Delta\,(c\omega)}.$$

Then $$GdF = \tfrac{1}{2}G_1 dF_1 + \tfrac{1}{2} \cdot \frac{c^2 \sin\omega \cos\omega d\omega}{1 - c^2 \sin^2\omega}.$$

Integrate: $$\therefore\ \daleth - \tfrac{1}{2} \cdot \daleth_1 = -\tfrac{1}{4}\log(1 - c^2\sin^2\omega) = -\tfrac{1}{2}\log\Delta.$$

This is the equation of reduction. By its repetition, you get

$$\daleth - 2^{-n}\daleth_n = -\tfrac{1}{2}\log\Delta - \tfrac{1}{4}\log\Delta_1 - \tfrac{1}{8}\log\Delta_2 - \ldots 2^{-n}\log\Delta_{n-1}.$$

When $n = \infty$, $\daleth = -2^{-1}\log\Delta - 2^{-2}\log\Delta_1 - 2^{-3}\log\Delta_2 - \text{\&c.}$ *ad infin.*

10. Thus far we have supposed $c\,c_1\,c_2 \ldots$ to converge rapidly. Only when c closely approaches 1 it is expedient to work backwards by $c\,c'\,c''$. Let ω' be so taken that $\sin(2\omega' - \omega) = c\sin\omega$, and ω'' so that $\sin(2\omega'' - \omega') = c'\sin\omega'$, &c. *We have a right to assume* ω *between* 0 *and* $\frac{1}{2}\pi$. Then $c\sin\omega$ being positive, so is $2\omega' - \omega$; and since $\sin(2\omega' - \omega) < \sin\omega$, $2\omega' - \omega$ is $< \omega$, or $\omega' < \omega$. Thus $\omega\,\omega'\,\omega''\,\omega''' \ldots$ a steadily *decreasing* series. But since $\tan(\omega - \omega') = b'\tan\omega'$, and the series $b\,b'\,b'' \ldots$ rapidly diminishes, the differences $\omega - \omega'$, $\omega' - \omega''$, $\omega'' - \omega''' \ldots$ rapidly diminish. Thus the series tends to a fixed limit $\dot\omega$. The greater is c, the more quickly is this limit sensibly attained.

With $$\frac{F}{C} = \tfrac{1}{2} \cdot \frac{F_1}{C_1} \text{ combine } \frac{B}{C} = \tfrac{1}{2} \cdot \frac{B}{C};$$

thence you have $$\frac{F}{B} = \frac{F_1}{B_1}.$$

Hence in reverse order $$\frac{F}{B} = \frac{F'}{B'} = \frac{F''}{B''} = \ldots = \frac{F^{(n)}}{B^{(n)}}.$$

When $b^{(n)2}$ is insignificant, $B^{(n)} = 1$, and $F^{(n)}$ becomes $\int_0 \sec\dot\omega d\dot\omega$ if $\dot\omega$ is the limit of $\omega\,\omega'\,\omega''\,\omega''' \ldots$ Hence $F = B \cdot \log\tan(45^\circ + \frac{1}{2}\dot\omega)$.

Also $$F = Cx, \quad \therefore\ \tfrac{1}{2}\pi x = \rho \cdot \log\tan(45^\circ + \tfrac{1}{2}\dot\omega).$$

11. When c is large, we proceed for E and G through J of Chap. III. Art. 11. We had $J = G + \dfrac{x}{F_b}$;

$$\therefore\ C(J - G) = \frac{Cx}{\frac{1}{2}\pi B}.$$

Similarly $$C_1(J_1 - G_1) = \frac{C_1 x_1}{\frac{1}{2}\pi B_1}.$$

But $$x_1 = 2x;\ \frac{B}{C} = \tfrac{1}{2} \cdot \frac{B_1}{C_1};\ \text{(Chap. II. Art. 13).}$$

$$\therefore\ \frac{C}{B} \cdot x = \frac{C_1}{B_1} \cdot x_1,$$

and $$C(J-G)=C_1(J_1-G_1),$$
whence $$CJ-C_1J_1=CG-C_1G_1.$$
Now we had $$CG-C_1G_1=c_1C_1\sin\omega_1 \text{ (Art. 8)},$$
$$\therefore\ CJ.-C_1J_1=c_1C_1\sin\omega_1,$$
or reversing the scale $C'J'-CJ=cC\sin\omega$.

But (Chap. III. Art. 12) when b^4 is omissible,
$$J=(1-\tfrac{1}{2}b^2)\sin\omega,$$
or $$(\sin\omega-J)=\tfrac{1}{2}b^2\sin\omega.$$

Put R for $C(\sin\omega-J)$, *which will vanish with* b, because it tends to $C.\tfrac{1}{2}b^2\sin\omega$, and C to $\log\frac{4}{b}$ only, while $b^2.\log b$ is evanescent.

Now $$cC=C'-C,\ \therefore\ C'J'-CJ=(C'-C)\sin\omega;$$
also $$R-R'=C(\sin\omega-J)-C'(\sin\omega'-J')$$
$$=(C'J'-CJ)-C'\sin\omega'+C\sin\omega.$$

Eliminate $C'J'-CJ$, then $R-R'=C'$, $(\sin\omega-\sin\omega')$. This is our new equation of reduction.

In it assume for $c\omega$ in succession $c'\omega'$, $c''\omega''$, ... $c^{(n)}\omega^{(n)}$, and add the results, $\therefore\ R-R^{(n)}=C'(\sin\omega-\sin\omega')+C''(\sin\omega'-\sin\omega'')+\ldots$
$$+C^{(n)}(\sin\omega^{(n-1)}-\sin\omega^{(n)}),$$
when $n=\infty$, $b^{(n)}=0$, $\therefore\ R^{(n)}=0$. Restore $C(\sin\omega-J)$ for R; then
$$CJ(c\omega)=C\sin\omega-C'(\sin\omega-\sin\omega')-C''(\sin\omega'-\sin\omega'')-\&c.$$

When c is at all near to 1, $\omega\,\omega'\,\omega''\ldots$ converge to $\grave{\omega}$ very rapidly, and $C'C''\ldots$ converge toward $\log\frac{4}{b'}$, but the smallness of b' does not here damage convergence. In fact we may replace the series $C\,C'\,C''\,C'''\ldots$ by their *proportionals* B, $2B'$, $2^2B''$, $2^3B'''\ldots$ and we know that $B\,B'\,B''\,B'''\ldots$ tend to 1, while the differences
$$\omega-\omega',\ \omega'-\omega'',\ \omega''-\omega''',\ldots$$
lessen far more rapidly than by the rate of successive halving.

From J we can deduce G and E at pleasure.

12. It remains to find the Diplonome, when c is too large for the process of Art. 9. By Chap. III. Art. 15 we put for a moment

$$K = \daleth + \frac{x^2}{2\rho}.$$

Also we have $$x_1 = 2x,\ \rho_1 = 2\rho,$$

$$\therefore\ \frac{x^2}{2\rho} = \tfrac{1}{2}\cdot\frac{x_1^{\ 2}}{2\rho_1} \text{ or } K - \daleth = \tfrac{1}{2}(K_1 - \daleth_1).$$

Hence $K - \frac{1}{2}K_1 = \daleth - \frac{1}{2}\daleth_1$. But the last by Art. 9 above

$$= -\tfrac{1}{2}\log\Delta.$$

For a moment let L mean $K + \log\Delta$, L' mean $K' + \log\Delta'$.

Then $$L - 2L' = (K + \log\Delta) - 2(K' + \log\Delta')$$
$$= (K - 2K') + (\log\Delta - 2\log\Delta').$$

Now by reversing the scale in $K - \frac{1}{2}K_1 = -\frac{1}{2}\log\Delta$, we find $2K' - K = -\log\Delta'$, whence

$$L - 2L' = \log\Delta' + (\log\Delta - 2\log\Delta') = \log\frac{\Delta}{\Delta'}.$$

This is our formula of repetition, and if repeated $(n-1)$ times by changing ω to $\omega'\ \omega''\ \omega'''\ldots$ it visibly yields

$$L - 2^n L^{(n)} = \log\frac{\Delta}{\Delta'} + 2\log\frac{\Delta'}{\Delta''} + \ldots + 2^{n-1}\log\frac{\Delta^{(n-1)}}{\Delta^{(n)}}.$$

When we proceed to make n infinite, the formidable multipliers 2^n, 2^{n+1} require close attention. Let us compute L or $K + \log\Delta$ when b^4, $b^6\ldots$ vanish, that is, all beyond b^2. Now

$$K = \int_0 J dF = \int_0 (1 - \tfrac{1}{2}b^2)\sin\omega,\ \frac{d\omega}{\Delta}.$$

But $$\Delta = \sqrt{(\cos^2\omega + b^2\sin^2\omega)},$$

$$\therefore\ K = \int_0 (1 - \tfrac{1}{2}b^2)\frac{\sin\omega}{\cos\omega}\cdot\frac{d\omega}{\sqrt{(1 + b^2\tan^2\omega)}}.$$

At present we seek only to estimate $2^n L^{(n)}$, and suppose $b^4\, b^6\ldots$ and higher powers of b^2 to be insignificant,

then $$K = \int_0 (1 - \tfrac{1}{2}b^2)(1 - \tfrac{1}{2}b^2\tan^2\omega)\tan\omega\, d\omega.$$

The product here of the binomials is

$$1 - \tfrac{1}{2}b^2(1 + \tan^2\omega) + \tfrac{1}{4}b^4\tan^2\omega$$

but we omit b^4 and retain only $1 - \frac{1}{2}b^2\sec^2\omega$.

Then observing that $\sec^2\omega \,.\, d\omega = d\tan\omega$, we obtain

$$K = \int_0 (\tan\omega\, d\omega - \tfrac{1}{2}b^2 \tan\omega \,.\, d\tan\omega)$$
$$= -\log\cos\omega - \tfrac{1}{4}b^2\tan^2\omega.$$

But $$\log\Delta = \log\cos\omega + \tfrac{1}{2}\log(1 + b^2\tan^2\omega)$$
$$= \log\cos\omega + \tfrac{1}{2}b^2\tan^2\omega, \text{ by developing the } log,$$

thence L or $K + \log\Delta = \frac{1}{4}b^2\tan^2\omega$, if $\tan\omega$ be finite and b^4 insignificant. Pass now from b to $b'\,b''\ldots b^{(n)}$, until $b^{(n)4}$ is omissible. Accompanying this change, $\omega\,\omega'\omega''\ldots\omega^{(n)}$ rapidly converge to $\dot\omega$ a fixed arc less than $\frac{1}{2}\pi$. Thus 2^nL^n converges to $\frac{1}{4}(2^nb^{(n)2})\tan^2\dot\omega$ and since the convergence of $b\,b'\,b''\,b^{(n)}$ is vastly more precipitate than that of $1, 2^{-1}, 2^{-2}\ldots 2^{n-1}$, there is no longer doubt that $(L - 2^nL^{(n)})$ converges to simple L.

Hence first $$L = \log\frac{\Delta}{\Delta'} + 2\log\frac{\Delta'}{\Delta''} + 2^2\log\frac{\Delta''}{\Delta'''} + \&c.$$
subtract $\log\Delta$ and you obtain K, or its equivalent

$$\left(\daleth(c\omega) - \frac{x^2}{2\rho}\right) = \log\frac{1}{\Delta'} + 2\log\frac{\Delta'}{\Delta''} + 2^2\log\frac{\Delta''}{\Delta'''} + \&c.$$

This converges excellently, so long as the terms are duly kept apart.

Scale of Gauss.

13. Gauss alighted on a scale of Index 2, *obverse* to that of Lagrange.

Suppose as with Lagrange $F(c_1\omega_1) = (1 + b)F(c\omega)$.

Let $$\sin\omega = \sqrt{-1}\tan\psi, \quad \sin\omega_1 = \sqrt{-1}\tan\psi_1,$$
$$\therefore F(c\omega) = \sqrt{-1}F(b\psi);\quad F(c_1\omega_1) = \sqrt{-1}F(b_1\psi_1),$$
whence $$F(b_1\psi_1) = (1 + b)F(b\psi).$$

Also $$\tan\omega_1 = \frac{(1+b)\tan\omega}{1 - b\tan^2\omega} \text{ becomes } \sin\psi_1 = \frac{(1+b)\sin\psi}{1 + b\sin^2\psi};$$
whence $$1 \pm \sin\psi_1 = \frac{(1 \pm \sin\psi)(1 \pm b\sin\psi)}{1 + b\sin^2\psi};$$
which shows that $\sin\psi_1$ and $\sin\psi$ reach 1 together, also reach -1 together; having begun together from zero. Thus ψ and ψ_1 *coincide at the end of every quadrant,* which fundamentally distinguishes the scale from Lagrange's.

Also since $B_1 = (1 + b)B$, we find $\dfrac{F(b_1\psi_1)}{B_1} = \dfrac{F(b\psi)}{B}$, or *the Mesonomes are equal.*

A second peculiarity is in the Conjugates. Let $\psi\psi^0$, $\psi_1\psi_1^0$ be conjugates. Then from the last equation we get (since $Fb = \frac{1}{2}\pi B$)

$$\frac{\frac{1}{2}\pi B_1 . - F(b_1\psi_1^0)}{B_1} = \frac{\frac{1}{2}\pi B - F(b\psi^0)}{B},$$

which at once shows $\dfrac{F(b_1\psi_1^0)}{B_1} = \dfrac{F(b\psi^0)}{B}$; or the Nomes *conjugates* to $\psi_1\psi$ have the same relation as the Nomes *belonging* to $\psi_1\psi$.

We easily obtain by combining $1 \pm \sin\psi_1$,

$$1 - \sin^2\psi_1 = \frac{(1-\sin^2\psi)(1-b^2\sin^2\psi)}{(1+b\sin^2\psi)^2},$$

whence
$$\cos\psi_1 = \frac{\cos\psi . \Delta(b\psi)}{1+b\sin^2\psi},$$

$$\tan\psi_1 = \frac{(1+b)\sin\psi}{\cos\psi . \Delta(b\psi)} = \frac{(1+b)\tan\psi}{\Delta(b\psi)};$$

$$(1+b)\cot\psi_1 = \cot\psi . \Delta(b\psi).$$

Again for $1 - b_1^2\sin^2\psi_1$; $\quad 1 + b = \dfrac{2\sqrt{b}}{b_1}$;

$$\therefore b_1\sin\psi_1 = \frac{2\sqrt{b} . \sin\psi}{1+b\sin^2\psi}; \quad \therefore 1 \pm b_1\sin\psi_1 = \frac{(1\pm\sqrt{b}\sin\psi)^2}{1+b\sin^2\psi};$$

hence
$$1 - b_1^2\sin^2\psi_1 = \left(\frac{1-b\sin^2\psi}{1+b\sin^2\psi}\right)^2,$$

and
$$\Delta(b_1\psi_1) = \frac{1-b\sin^2\psi}{1+b\sin^2\psi}.$$

In reverse $\quad b\sin^2\psi = \dfrac{1-\Delta_1}{1+\Delta_1}$, $\sqrt{b}\sin\psi = \sqrt{\dfrac{1-\Delta_1}{1+\Delta_1}}$.

To calculate this scale for the 3rd Elliptic Integral Legendre puts $\sin\lambda = b\sin\psi$, and generally $\sin\lambda_n = b_n\sin\psi_n$.

$$\therefore \cos\lambda = \Delta(b\psi), \quad \cos\lambda_n = \Delta_n;$$

but we had
$$\sqrt{b}\sin\psi = \sqrt{\frac{1-\Delta_1}{1+\Delta_1}},$$

therefore
$$= \sqrt{\frac{1-\cos\lambda_1}{1+\cos\lambda_1}} = \tan\tfrac{1}{2}\lambda_1.$$

Multiply by $\sqrt{b}$, $\therefore b\sin\psi = \sqrt{b}\tan\frac{1}{2}\lambda_1$ or $\sin\lambda = \sqrt{b} . \tan(\frac{1}{2}\lambda_1)$. This is the new law of succession for $\lambda_1\lambda_2\ldots\lambda_n$. Since $bb_1b_2b_3\ldots$ converge to $1\ldots$, the arcs $\lambda\lambda_1\lambda_2\ldots$ tend to $\frac{1}{2}\pi$, making at last

$$\sin\lambda_n = \tan . \tfrac{1}{2}\lambda_{n+1}, \text{ when } \lambda_n = \lambda_{n+1} = 90^\circ.$$

This scale sometimes affords useful transformations; but it is of very inferior importance to Lagrange's.

CHAPTER VI.

SCALE OF LEGENDRE WITH INDEX 3.

1. IN Lagrange's scale we worked from $t = \tan\omega$,

$$F = \frac{dt}{\sqrt{(1+t^2)}\sqrt{(1+b^2t^2)}}.$$

It is now preferable to start from

$$F = \frac{dv}{\sqrt{(1-v^2)}\sqrt{(1-c^2v^2)}}.$$

If in this form we change v into $c^{-1}v^{-1}$, the result is first

$$\frac{c^{-1}dv^{-1}}{\sqrt{(1-c^{-2}v^{-2})}\sqrt{(1-v^{-2})}};\ \text{next}\ \frac{-dv}{\sqrt{(c^2v^2-1)}\sqrt{(v^2-1)}};$$

and if further we multiply the denominator by $\sqrt{-1}\,.\sqrt{-1}$ and the numerator by the equivalent -1, we reproduce the original F.

If then, in hope of deducing $F(c\omega) = \mu F(h, \theta)$, we *tentatively* assume $\mu z = v\,.\,\dfrac{1-mv^2}{1-nv^2}$, in which $v = \sin\omega$, $z = \sin\theta$, and m, n, μ are disposable constants, we at once discern that the change of v into $c^{-1}v^{-1}$ simultaneously with that of z into $h^{-1}z^{-1}$ must be admissible; also that as v and z begin from zero together so must v^{-1} and z^{-1}; that is, v and z must be infinite together. But the latter condition is already provided in the equation $\mu z = v\,.\,\dfrac{1-mv^2}{1-nv^2}$, which converges to $\mu\,.\,\dfrac{z}{v} = \dfrac{m}{n}$, when v and z are infinite.

2. In $\mu z = v\,.\,\dfrac{1-mv^2}{1-nv^2}$ introduce $c^{-1}v^{-1}$ for v and $h^{-1}z^{-1}$ for z, and

it becomes
$$\frac{\mu}{hz} = \frac{1}{cv}\,.\,\frac{c^2v^2-m}{c^2v^2-n}.$$

Invert both;
$$\mu^{-1}hz = cv\,.\,\frac{n-c^2v^2}{m-c^2v^2}.$$

This ought to be made *identical with* the previous equation. For this we need two conditions; first

$$(1) \qquad c^2 = mn, \ \therefore \ \frac{n - c^2v^2}{m - c^2v^2} = \frac{n(1 - mv^2)}{m(1 - nv^2)};$$

which makes the newer equation

$$\mu^{-1}hz = cv \cdot \frac{n}{m} \cdot \frac{1 - mv^2}{1 - nv^2}.$$

Divide the original by this last,

$$\therefore \ \frac{\mu^2}{h} = \frac{m}{nc}.$$

$$(2) \qquad h = \mu^2 \cdot \frac{nc}{m} = \mu^2 \cdot \frac{mnc}{m^2} = \mu^2 \cdot \frac{c^3}{m^2}.$$

Two assumptions concerning the five constants have thus been made. *Two more* are allowable.

3. As a *third*, suppose θ to become $\frac{3}{2}\pi$, when $\omega = \frac{1}{2}\pi$; then for $v = 1$, we have $z = -1$, or from the original equation $\mu = \dfrac{m-1}{1-n}$.

Eliminate μ; $\qquad \therefore \ z = \dfrac{1-n}{m-1} \cdot \dfrac{v - mv^3}{1 - nv^2}$.

We know that when $1 - v = 0$, $1 + z = 0$ by our third hypothesis; and here $1 + z =$ a fraction in terms of v, whose numerator

$$\overline{m-1} \cdot (1 - nv^2) + (1 - n)(v - mv^3),$$

must have $(1 - v)$ as a factor.

We may therefore assume $(1 - v)(p + qv + rv^2)$ identical with

$$(m-1)(1 - nv^2) + (1 - n)(v - mv^3).$$

First then from $v = 0$, $p = m - 1$, and from $v = \infty$,

$$r = (1 - n)\, m = m - c^2.$$

Next, $\qquad q - p = 1 - n$, or $q = (m - 1) + (1 - n) = m - n$.

Also, $\qquad q - r = (m - 1)\, n = c^2 - n$.

Eliminate r; $\ \therefore \ q = (m - c^2) + (c^2 - n) = m - n$ *as before*.

Thus the 3 $(p\,q\,r)$ rightly fulfil the 4 equations and we attain

$$1 + z = (1 - v) \cdot \frac{(m-1) + (m-n)\,v + (m - c^2)\,v^2}{(m-1)(1 - nv^2)}.$$

As a *fourth* and final assumption, let the numerator of the last fraction *be an algebraic square.*

This requires $(m-1)(m-c^2)=\frac{1}{4}(m-n)^2$,

$$\therefore \frac{1+z}{1-v}=\frac{\left(1+v\sqrt{\frac{r}{p}}\right)^2}{1-nv^2}.$$

4. To connect the two equations, $mn=c^2$ ....(1),

$$m-n=2\sqrt{(m-1)}\sqrt{(m-c^2)},$$

or $$1-m^{-1}n=2\sqrt{(1-m^{-1})}\sqrt{(1-c^2m^{-1})}\text{..............(2)}.$$

We fall back on our Trisection in Ch. IV. Art. 9, and assume $\sin^2\beta=m^{-1}$ which for $mn=c^2$ gives $n=c^2\sin^2\beta$ and for equation (2)

$$1-c^2\sin^4\beta=2\cos\beta\sqrt{(1-c^2\sin^2\beta)}.$$

But in Ch. IV. Art. 3, making $\omega=\theta=\beta$ and giving to V its value $(1-c^2\sin^4\beta)^{-1}$,

$$\sin\eta=\frac{2\sin\beta\cos\beta\,.\,\Delta(c\beta)}{1-c^2\sin^4\beta};$$

whence we get $$\frac{\sin\eta}{\sin\beta}=1,$$

though $F(c\eta)=2F(c\beta)$.

We cannot infer $\eta=\beta$ without making each vanish: therefore we solve by $\eta=\pi-\beta$; and deduce $F(c,\eta)=F(c,\pi-\beta)$, which is $2F_c-F(c\beta)$. But since also $F(c\eta)=2F(c\beta)$, we find $F(c\beta)=\frac{2}{3}F_c$.

5. Next assume α such that $F(c\alpha)=\frac{1}{3}F_c$; then α and β are Conjugate.

We had $$\sqrt{\frac{r}{p}}=\sqrt{\frac{m-c^2}{m-1}}=\sqrt{\frac{1-c^2m^{-1}}{1-m^{-1}}}$$

$$=\frac{\sqrt{(1-c^2\sin^2\beta)}}{\sqrt{(1-\sin^2\beta)}}=\frac{\Delta(c\beta)}{\cos\beta}=\frac{1}{\sin\alpha}.$$

Also $$\frac{1+z}{1-v}=\frac{(1+\sqrt{rp^{-1}}\,.\,v)^2}{1-nv^2}=\frac{(1+v\operatorname{cosec}\alpha)^2}{1-c^2v^2\sin^2\beta},$$

but for conciseness we may keep n and write A for cosec α.

Then $$\frac{1+z}{1-v}=\frac{(1+Av)^2}{1-nv^2}.$$

Change v to $-v$, ω to $-\omega$, θ to $-\theta$, z to $-z$, so that

$$\frac{1-z}{1+v}=\frac{(1-Av)^2}{1-nv^2}.$$

Multiply together the two last, and take $\sqrt{}$.

$$\therefore \sqrt{\frac{1-z^2}{1-v^2}} \text{ or } \frac{\cos\theta}{\cos\omega} = \frac{1-A^2v^2}{1-nv^2}.$$

6. We have provided that to change v into $(cv)^{-1}$ shall change z into $(hz)^{-1}$. Effect this change in the last, and it yields

$$\frac{cv}{hz}\sqrt{\frac{1-h^2z^2}{1-c^2v^2}} = \frac{A^2}{n}\cdot\frac{1-A^{-2}c^2v^2}{1-n^{-1}c^2v^2}.$$

But in the last denominator $c^2n^{-1}=m$. Multiply by

$$\frac{\mu z}{v} = \frac{1-mv^2}{1-nv^2},$$

$$\therefore \mu\cdot\frac{c}{h}\cdot\frac{\Delta(h\theta)}{\Delta(c\omega)} = \frac{A^2}{n}\cdot\frac{1-A^{-2}c^2v^2}{1-nv^2}.$$

Make $v=0,\ \theta=0$; $\qquad \therefore \mu\cdot\frac{c}{h} = \frac{A^2}{n}$,

hence finally $\qquad \dfrac{\Delta(h\theta)}{\Delta(c\omega)} = \dfrac{1-c^2v^2\sin^2\alpha}{1-c^2v^2\sin^2\beta}.$

Cor. The last equation of constants shows

$$h = \mu\cdot\frac{nc}{A^2} = \frac{m-1}{1-n}\cdot nc\sin^2\alpha,$$

or

$$h = \frac{1-m^{-1}}{1-n}\cdot mnc\sin^2\alpha = c^3\sin^2\alpha\cdot\frac{1-\sin^2\beta}{1-c^4\sin^2\beta}.$$

The square root of the last fraction is $\dfrac{\cos\beta}{\Delta(c\beta)}$ or $=\sin\alpha$, by law of Conjugates.

Finally, then $\qquad h = c^3\sin^4\alpha$;

an equation cardinal in these constants.

Contrast it with $h=\left(\dfrac{c}{1+b}\right)^2$ of Landen's scale.

7. Again for conciseness, write f for $c\sin\alpha$; then since A means $(\sin\alpha)^{-1}$ we have $A\,.f=c$. We have obtained three chief equations

$$\mu z = \frac{v-mv^3}{1-nv^2};\qquad \sqrt{\frac{1-z^2}{1-v^2}} = \frac{1-A^2v^2}{1-nv^2};\qquad \frac{\Delta_1}{\Delta} = \frac{1-f^2v^2}{1-n^2}.$$

Multiply the two last together; then

$$\frac{\Delta_1}{\Delta} \cdot \sqrt{\frac{1-z^2}{1-v^2}} = \frac{1-(A^2+f^2)\,v^2+A^2f^2v^4}{(1-nv^2)^2}.$$

But differentiate the first of the three; then

$$\mu \cdot \frac{dz}{dv} = \frac{1-(3m-n)\,v^2+mnv^4}{(1-nv^2)^2}.$$

Compare the numerators of the two last fractions on the right. They have the same coefficient of v^4; for

$$A^2f^2 = c^2 = mn.$$

Hence the two fractions will be identical *if* we can establish that

$$3m - n = A^2 + f^2.$$

Now, *I say*, this is true, by the law of Trisection, Ch. IV., Art. 9.

There we have k for $\sin\alpha$,

$$1-k=\cos\beta, \quad \sin^2\beta = 2k-k^2.$$

The equation to be here *tested* is

$$3\operatorname{cosec}^2\beta - c^2\sin^2\beta = \operatorname{cosec}^2\alpha + c^2\sin^2\alpha.$$

Expressed in functions of k it becomes

$$\frac{3}{2k-k^2} - c^2\,(2k-k^2) = \frac{1}{k^2} + c^2k^2.$$

Expunge c^2k^2 from both sides, then solve for c^2; you obtain

$$c^2 = \frac{2k-1}{k^3\,.\,(2-k)}; \text{ *precisely correct.*}$$

We have then proved that

$$\mu \cdot \frac{dz}{dv} = \frac{\Delta_1}{\Delta} \cdot \sqrt{\frac{1-z^2}{1-v^2}},$$

or $$\frac{\mu dz}{\Delta_1 . \sqrt{(1-z^2)}} = \frac{dv}{\Delta . \sqrt{(1-v^2)}}; \text{ or } \mu F(h,\ \theta) = F(c\omega).$$

COR. Since when

$$\omega = \tfrac{1}{2}\pi, \quad \theta = \tfrac{3}{2}\pi, \quad \mu\,.\,3F_h = F_c$$

Eliminating μ,

$$\frac{F(c\omega)}{F_c} = \frac{1}{3} \cdot \frac{F(h\theta)}{F_h};$$

and 3 is the Index of the new scale.

8. An Obverse scale is hereby found, as that of Gauss from Lagrange's; but it is perhaps of less importance since Jacobi's generalization.

Writing $$\sin\omega = \sqrt{-1}\tan\psi,$$

and $$\sin\theta = \sqrt{-1}\tan\chi,$$

with $$h^2 + g^2 = 1,$$

you find by usual routine

$$F(c\omega) = \sqrt{-1}\,.\,F(b\psi),\quad F(h\theta) = \sqrt{-1}\,F(g\chi),$$

whence $$F(b\psi) = \mu F(g\chi).$$

But since the primary equation is now transformed into

$$\mu\,.\,\frac{\tan\chi}{\tan\psi} = \frac{1 + A^2\tan^2\psi}{1 + n\tan^2\psi},$$

ψ and χ begin together from zero, and when $\tan\psi$ becomes infinite,

$$\mu\tan\chi = A^2\,.\,n^{-1}\,.\tan\psi,\quad \text{or}\quad \tan\chi \text{ is also infinite.}$$

In short, they arrive together at the end of every quadrant. Making

$$\psi = \tfrac{1}{2}\pi,\quad \chi = \tfrac{1}{2}\pi,\quad F_b = \mu F_g\,.$$

The analogy of this Obverse to Gauss's is perfect.

9. *Here* of the constants $c\,h\,\mu$ *only one* is arbitrary; $\alpha\beta$ are deduced from c by Trisection. Legendre finds it best to refer all to μ, which he names the *Regulator*. In Art. 3

$$\mu = \frac{m-1}{1-n} = \frac{\cot^2\beta}{1 - c^2\sin^2\beta},$$

but $$\frac{\cos\beta}{\Delta(c\beta)} = \sin\alpha,\ \text{by law of conjugates;}$$

$$\therefore\quad \mu = \frac{\sin^2\alpha}{\sin^2\beta}\,.$$

As we began from $v = \sin\omega,\quad z = \sin\theta,$

the primary differential equation is

$$\frac{\mu d\theta}{\Delta(h\theta)} = \frac{d\omega}{\Delta(c\omega)}\ \text{or}\ \mu\frac{d\theta}{d\omega} = \frac{\Delta(h,\theta)}{\Delta(c\omega)}\,.$$

But in Art. 6 we found the last fraction

$$=\frac{1-c^2v^2\sin^2\alpha}{1-c^2v^2\sin^2\beta};$$

$$\therefore\quad \mu\frac{d\theta}{d\omega}=\frac{1-c^2v^2\sin^2\alpha}{1-nv^2},$$

which admits of new integration.

Since $$\mu=\frac{\sin^2\alpha}{\sin^2\beta};$$

$$\therefore\quad \left.\begin{aligned}c^2\sin^2\alpha&=\mu c^2\sin^2\beta\\&=\mu n\end{aligned}\right\},$$

and $$\frac{d\theta}{d\omega}=\frac{\mu^{-1}-nv^2}{1-nv^2}$$

$$=1+\frac{\mu^{-1}-1}{1-nv^2}.$$

Restore $$v^2=\sin^2\omega.$$

Integrate: first, $$\theta=\omega+(\mu^{-1}-1)\int\frac{d\omega}{1-n\sin^2\omega}.$$

Here

$$\mu^{-1}-1=\frac{\sin^2\beta}{\sin^2\alpha}-1=\frac{2k-k^2}{k^2}-1=\frac{2(1-k)}{k}=\frac{2\cos\beta}{\sin\alpha}$$

which, by Conjugate Law, $=2\Delta(c\beta)$.

Thus $$\frac{\theta-\omega}{2}=\int_0\frac{\Delta(c\beta)\,d\omega}{1-c^2\sin^2\beta\,.\,\sin^2\omega}.$$

In the denominator here

$$c^2\sin^2\beta=1-\Delta^2(c\beta).$$

For a moment write $$e=\Delta(c\beta).$$

The denominator is now

$$1-(1-e^2)\sin^2\omega=\cos^2\omega+e^2\sin^2\omega;$$

$$\therefore\quad \frac{\theta-\omega}{2}=\int_0\frac{e\,d\omega}{\cos^2\omega+e^2\sin^2\omega}=\int_0\frac{e\,.\sec^2\omega d\omega}{1+e^2\tan^2\omega}=\int_0\frac{ed\,.\tan\omega}{1+e^2\tan^2\omega}.$$

Our integral is now $$\frac{\theta-\omega}{2}=\tan^{-1}.(e\tan\omega).$$

Pass to tangent on both sides; then $\tan\frac{1}{2}(\theta-\omega)=\Delta(c\beta).\tan\omega$.

This is the equation by which θ is computed from given ω in Legendre's scale, but *at each step* $\Delta(c\beta)$ must be found from the

modulus, as first from c; at the second step h must first be calculated by some equivalent to $\qquad h = c^8 \sin^4 \alpha,$
and so on.

Such complexity in the constant makes this scale (though its convergence is far better than Lagrange's) practically unavailing.

This weakness is overcome in the Higher Theory if we pass from c to ρ as leading constant. A table of Trisection with ω as argument would do much, or would have done.

10. Another relation is found from Art. 5. We had

$$\frac{1-z}{1+v} = \frac{(1-Av)^2}{1-nv^2},$$

whence

$$\frac{1+z}{1-v} = \frac{(1+Av)^2}{1-nv^2}.$$

Eliminate $1 - nv^2$, then

$$\sqrt{\frac{1+z}{1-z}} = \frac{1+Av}{1-Av} \cdot \sqrt{\frac{1-v}{1+v}} = \frac{\sin\alpha + v}{\sin\alpha - v}\sqrt{\frac{1-v}{1+v}},$$

or

$$\tan(45^\circ + \tfrac{1}{2}\theta) = \frac{\sin\alpha + \sin\omega}{\sin\alpha - \sin\omega}, \ \tan(45^\circ - \tfrac{1}{2}\omega),$$

where the fraction admits the form

$$\frac{\tan\frac{1}{2}(\alpha+\omega)}{\tan\frac{1}{2}(\alpha-\omega)}.$$

This gives θ from ω, if we are expert in trisecting for α.

11. The constants, as referred to μ the Regulator, which is $> \frac{1}{3}$.

First, as Definition, k means $\sin\alpha$, where

$$F(c, \alpha) = \tfrac{1}{3}F_c; \quad \text{also } F(c, \beta) = \tfrac{2}{3}F_c.$$

Then $c^2 = \dfrac{2k-1}{k^3(2-k)}$ is test of Trisection; α and β are Conjugate.

Therefore

$$\sin\beta = \frac{\cos\alpha}{\Delta(c\alpha)}.$$

Also since

$$F(c\beta) = 2F(c\alpha);$$

$$\therefore \quad \tan\tfrac{1}{2}\beta = \tan\alpha \,.\, \Delta(c\alpha).$$

To eliminate c out of the two last, we eliminate $\Delta(c\alpha)$, which gives

$$\sin\beta\tan\tfrac{1}{2}\beta = \cos\alpha \,.\, \tan\alpha = \sin\alpha.$$

But

$$\sin\beta \,.\, \tan\tfrac{1}{2}\beta = \sin\beta\sqrt{\frac{1-\cos\beta}{1+\cos\beta}} = 1 - \cos\beta.$$

Finally then $\sin\alpha + \cos\beta = 1$,

a relation *independent* of c.

Since $\sin\alpha = k, \quad \cos\beta = 1 - k,$

$$\therefore \quad \sin^2\beta = 2k - k^2, \quad 2k - 1 = \sin\alpha - \cos\beta,$$

$$\mu = \frac{\sin^2\alpha}{\sin^2\beta} = \frac{k^2}{2k-k^2} = \frac{k}{2-k},$$

whence $$k \text{ or } \sin\alpha = \frac{2\mu}{1+\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

also $$\cos\beta = 1 - k = \frac{1-\mu}{1+\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

in reverse $$\mu = \frac{1-\cos\beta}{1+\cos\beta}, \quad \therefore \sqrt{\mu} = \tan\tfrac{1}{2}\beta \quad \ldots\ldots\ldots\ldots(3).$$

By squaring the penult $\cos^2\beta = \left(\frac{1-\mu}{1+\mu}\right)^2$,

so too $$\cos^2\alpha = (1-k)(1+k) = \frac{1-\mu}{1+\mu} \cdot \frac{1+3\mu}{1+\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$\Delta(c\beta) = \frac{\cos\beta}{\sin\alpha} = \frac{1-\mu}{2\mu}.$$

Further $$\sin^2\alpha = \frac{4\mu^2}{(1+\mu)^2}, \quad \sin^2\beta = \frac{\sin^2\alpha}{\mu} = \frac{4\mu}{(1+\mu)^2}.$$

In $$c^2 = \frac{\sin\alpha - \cos\beta}{k^2(2k-k^2)} = \frac{\sin\alpha - \cos\beta}{\sin^2\alpha \,.\, \sin^2\beta},$$

substitute in functions of μ; therefore

$$c^2 = \frac{\dfrac{2\mu}{1+\mu} - \dfrac{1-\mu}{1+\mu}}{\left(\dfrac{2\mu}{1+\mu}\right)^2 \cdot \dfrac{4\mu}{(1+\mu)^2}} = \frac{(3\mu-1)(1+\mu)^3}{4^2 \,.\, \mu^3} \quad \ldots\ldots\ldots(5).$$

Next, α and β being Conjugate,

$$\tan\alpha \,.\, \tan\beta \,.\, b = 1,$$

or $$b^2 = \frac{\cos^2\alpha}{\sin^2\alpha} \cdot \frac{\cos^2\beta}{\sin^2\beta},$$

or $$b^2 = \frac{\left(\dfrac{1-\mu}{1+\mu} \cdot \dfrac{1+3\mu}{1+\mu}\right)\left(\dfrac{1-\mu}{1+\mu}\right)^2}{\left(\dfrac{2\mu}{1+\mu}\right)^2 \cdot \dfrac{4\mu}{(1+\mu)^2}} = \frac{(1+3\mu)(1-\mu)^3}{4^2\mu^3} \quad \ldots\ldots(6).$$

Thus b^2 is the same function of $-\mu$, as is c^2 of $+\mu$.

12. Further

$$h = c^3 \sin^4 \alpha; \quad h^3 = \left(\frac{3\mu - 1 . \overline{1+\mu}|^3}{4^2\mu^3}\right)^3 . \left\{\frac{4\mu^2}{(1+\mu)^2}\right\}^4$$

$$= (3\mu - 1)^3 . \frac{1+\mu}{4^2 . \mu} \quad \text{.........................(7).}$$

But assume $3\mu\mu' = 1$, we find

$$h^2 = \left(\frac{1}{\mu'} - 1\right)^3 . \frac{\mu^{-1} + 1}{4^2} = (1 - \mu')^2 . \frac{3\mu' + 1}{4^2\mu'^3} .$$

Thus h^2 is the same function of μ' as c^2 was of μ. We infer that g^2 is the same function of $-\mu'$ as is h^2 of $+\mu'$ or

$$g = (1 + \mu')^3 . \frac{3\mu' - 1}{4^2\mu'^3} ,$$

or replacing μ' by $(3\mu)^{-1}$ we find $\mu < 1$, since

$$g^2 = (3\mu + 1)^3 . \frac{1 - \mu}{4^2\mu} \quad \text{.........................(8).}$$

Thus $c\,b\,h\,g$ are related as $g\,h\,b\,c$.

The same is denoted by $F_c = 3\mu . F_h$ and $\mu F_g = F_b$; whence

$$\frac{B}{C} = \frac{1}{3}\frac{G}{H},$$

whence further $\rho = \frac{1}{3} . \rho_1$, if ρ_1 is related to H, G as ρ to C, B.

13. We may now further estimate

$$\Delta^2 (c\alpha) = \frac{(3\mu + 1)(1 - \mu)}{4\mu} ,$$

$$c^2 \sin^2 \alpha = \frac{(3\mu - 1)(1 + \mu)}{4\mu}; \quad c^2 \sin^2 \beta = \frac{(3\mu - 1)(1 + \mu)}{4\mu^2};$$

$$c^2 . \cos^2 \alpha = (1 - \mu^2) . \frac{9\mu^2 - 1}{4^2\mu^3}; \quad c^2 \cos^2 \beta = (3\mu - 1)(1 - \mu^2) . \frac{1 - \mu}{4^2\mu^3}$$

14. To find $\dfrac{d\mu}{d\rho}$. We had $\dfrac{dc}{d\rho} = -b^2c . C^2$ in Ch. II.; therefore

$$\frac{d\mu}{d\rho} = \frac{d\mu}{dc} . \frac{dc}{d\rho};$$

and from $d . \log (c^2)$ you get

$$\frac{2}{3} . \frac{dc}{c} = \frac{(1 - \mu)}{(3\mu - 1)(1 + \mu)} . \frac{d\mu}{\mu} ,$$

so that $$\frac{d\mu}{dc} = \frac{2}{3}\frac{\mu}{c} . \frac{(3\mu - 1)(1 + \mu)}{(1 - \mu)^2} .$$

This yields $$\frac{d\mu}{\mu\, d\rho} = -\tfrac{2}{3} \,.\, C^2 b^2 \,.\, \frac{(3\mu - 1)(1+\mu)}{(1-\mu)^2},$$

in which if you give to b^2 its value as a function of μ, the result is

$$\frac{d\mu}{\mu\, d\rho} = -\tfrac{2}{3}\, C^2 c^2 \cos^2 \alpha.$$

15. To find the relation between $\aleph_c$ and $\aleph_h$ in this scale—(the Ancillae).

From equation (a)

$$b^2 c \,.\, \frac{dE_c}{dc} = E_c - b^2 F_c,$$

which also $$= F_c\,(\aleph_c - b^2).$$

Divide by F_c, $$\therefore\quad b^2 c \,.\, \frac{d \log F_c}{dc} = \aleph_c - b^2.$$

But $$-dc = b^2 c \,.\, C^2 d\rho.$$

Eliminate dc, $$\therefore\quad -\frac{d \log F_c}{d\rho} = C^2 \,.\, (\aleph_c - b^2).$$

Call this for a moment $\Gamma(c)$, or

$$-\frac{d \log F_c}{d\rho} = \Gamma(c).$$

Change $c\rho$ to h, 3ρ, $$\therefore\quad -\frac{d \log F_h}{3 d\rho} = \Gamma(h).$$

Now $F_c = 3\mu \,.\, F_h$; $$\therefore\quad \frac{d \log F_c}{d\rho} - \frac{d \log F_h}{d\rho} = \frac{d \log \mu}{d\rho},$$

i.e. $$\Gamma(c) - 3\Gamma(h) = -\frac{d\mu}{\mu d\rho} = \tfrac{2}{3} C^2 c^2 \cos^2 \alpha \quad \text{(Art. 13)}.$$

Restore the values of Γ and divide by $C^2 = 9\mu^2 H^2$,

$$\therefore\quad (\aleph_c - b^2) - \frac{1}{3\mu^2}(\aleph_h - g^2) = \tfrac{2}{3} c^2 \cos^2 \alpha.$$

Subtract the last from

$$c^2 - \frac{h^2}{3\mu^2} = \frac{(3\mu - 1)(1+p)}{48\mu^3} \,.\, \{3\,(1+\mu)^2 - (3\mu - 1)^2\},$$

observing that

$$\tfrac{2}{3} c^2 \cos^2 \alpha = \tfrac{2}{3} \,.\, \frac{(9\mu^2 - 1)(1 - \mu^2)}{16\mu^3}$$

$$= \frac{(3\mu - 1)(1+\mu)}{48\mu^3} \,.\, \{2\,(3\mu + 1)(1 - \mu)\};$$

$$\therefore \quad (1-\aleph_c) - \frac{1}{3\mu^2}(1-\aleph_h) = \frac{(3\mu-1)(1+\mu)}{48\mu^3} \cdot 8\mu$$

$$= \frac{(3\mu-1)(1+\mu)}{6\mu^2} = \tfrac{2}{3}c^2 \sin^2\beta \quad \text{(Art. 11)}.$$

This is the simplest relation, but cannot conduce to calculate $\aleph_c$ by repetition.

16. Problem. To connect $G(c\omega)$ and $G(h\theta)$ in this scale.

We had $\dfrac{dx}{d\rho} = CG(c\omega^0)$ when ω is *constant*. If ω and c vary together in x, we have $\quad \delta x = \dfrac{dx}{d\omega}\delta\omega + \dfrac{dx}{d\rho}\delta\rho = \dfrac{\delta\omega}{C\Delta\omega} + CG(c\omega^0)\,\delta\rho.$

Change from $c\omega$ to $h\theta$, which changes x to $3x$ and ρ to 3ρ,

$$\therefore \quad 3\delta x = \frac{\delta\theta}{H\Delta(h\theta)} + HG(h\theta^0)\,.\,3\delta\rho.$$

Identify x and ρ in these and make ω constant or $\delta\omega = 0$.

Then
$$CG(c\omega^0) - HG(h\theta^0) = \frac{1}{3H\Delta(h\theta)} \cdot \frac{d\theta}{d\rho}.$$

We need to find $\dfrac{d\theta}{d\rho}$ with ω constant.

Above we had $\frac{1}{2}(\theta-\omega) = \tan^{-1}\{\Delta(c\beta)\tan\omega\}$. Differentiate with ω constant, changing $\Delta(c\beta)$ to its equivalent in μ.

$$\therefore \quad \frac{d\theta}{d\mu} = \frac{d}{d\mu}\,.\,2\tan^{-1}.\left(\frac{1-\mu}{2\mu}\tan\omega\right) = \frac{-4\tan\omega}{4\mu^2 + (1-\mu^2)\tan^2\omega}.$$

Also
$$\frac{d\mu}{d\rho} = -\tfrac{2}{3}\,.\,C^2\,.\,\frac{(9\mu^2-1)(1-\mu^2)}{16\mu^2} \quad \text{(Art. 14)},$$

$$C = 3\mu H, \quad C^2 = 3\mu CH\,;$$

$$4\mu^2 + (1-\mu)^2\tan^2\omega = 4\mu^2\sec^2\omega\,\{\cos^2\omega + \Delta^2(c\beta)\sin^2\omega\}$$
$$= 4\mu^2\sec^2\omega\,.\,(1 - c^2\sin^2\beta\sin^2\omega),$$

$$\therefore \quad \frac{d\theta}{d\rho} = \frac{d\theta}{d\mu}\cdot\frac{d\mu}{d\rho} = \frac{4\tan\omega\,.\,2\mu CH(9\mu^2-1)(1-\mu^2)}{4\mu^2\sec^2\omega\,(1-c^2\sin^2\beta\sin^2\omega)\,.\,16\mu^2}$$

$$= 2CH\,.\,\frac{\sin\omega\cos\omega}{1-c^2\sin^2\beta\sin^2\omega}\,.\,(c^2\sin^2\beta)\,.\,\Delta^2(c\alpha)$$

which is to be divided by $3H\Delta(h,\theta)$. Observe that

$$\Delta(h\theta)(1 - c^2\sin^2\beta\sin^2\omega) = \Delta(c\omega)(1 - c^2\sin^2\alpha\,.\,\sin^2\omega),$$

$$\frac{1}{3H\,.\,\Delta(h,\theta)}\cdot\frac{d\theta}{d\rho}=\tfrac{2}{3}\sin\omega\sin\omega^0\,.\,\frac{c^2\sin^2\beta\,.\,\Delta^2(c\alpha)}{1-c^2\sin^2\alpha\,.\,\sin^2\omega};$$

so that $\quad CG(c\omega^0)-HG(h\theta^0)-\tfrac{2}{3}Cc^2\sin^2\beta\,.\,\dfrac{\Delta^2(c\alpha)\sin\omega\,.\,\sin\omega^0}{1-c^2\sin^2\alpha\sin^2\omega}.$

We proceed to change ω^0 to ω, and inquire how this will change θ.

$$F(c\omega^0)=F_c-F(c\omega)=3\mu F_h-\mu F(h,\theta)$$
$$=\mu\{2F_h+F(h\theta^0)\}=\mu\,.\,F(h,\pi+\theta^0).$$

Hence when ω changes to ω^0, θ changes to $\pi+\theta^0$, and θ^0 to $\pi-\theta$. But $G(\pi-\theta)=G(\theta)$, $\therefore$ we have simply

$$CG(c\omega)-HG(h\theta)=\tfrac{2}{3}Cc^2\sin^2\beta\,.\,\Delta^2(c\alpha)\,.\,\frac{\sin\omega^0\sin\omega}{1-c^2\sin^2\alpha\sin^2\omega^0}.$$

Finally in the denominator, write $\dfrac{\cos^2\omega}{\Delta^2(c\omega)}$ for $\sin^2\omega^0$. The last fraction

becomes $$\frac{\Delta^2(c\omega)\sin\omega^0\sin\omega}{\Delta^2(c\omega)-c^2\,.\,\sin^2\omega\cos^2\omega},$$

and the new denominator is

$$1-c^2\sin^2\omega-c^2\sin^2\alpha\,(1-\sin^2\omega)=(1-c^2\sin^2\alpha)-c^2\sin^2\omega\cos^2\alpha.$$

But $\cos^2\alpha$ by law of Conjugates $=\Delta^2(c\alpha)\,.\,\sin^2\beta$. Hence the last

$$=\Delta^2(c\alpha)\{1-c^2\sin^2\beta\,.\,\sin^2\omega\},$$

so that $$CG(c\omega)-HG(h\theta)=\tfrac{2}{3}C\,.\,\frac{c^2\sin^2\beta\,.\,\sin\omega\cos\omega\Delta(c\omega)}{1-c^2\sin^2\beta\sin^2\omega};$$

since $$\sin\omega^0=\frac{\cos\omega}{\Delta(c\omega)}.$$

17. The chief interest of this result is in its application to the Diplonome.

Multiply the last by $\dfrac{dF(c\omega)}{C}=\dfrac{1}{3}\cdot\dfrac{dF(h\theta)}{H}=\dfrac{1}{C}\cdot\dfrac{d\omega}{\Delta(c\omega)}$,

$$\therefore\quad G(c\omega)\,dF(c\omega)-\tfrac{1}{3}G(h,\theta)\,dF(h,\theta)=\frac{2}{3}\,\frac{c^2\sin^2\beta\,.\,\sin\omega\cos\omega d\omega}{1-c^2\sin^2\beta\sin^2\omega}.$$

Integrate: $\quad \daleth c\omega-\tfrac{1}{3}\,.\,\daleth(h\theta)=-\tfrac{1}{3}\log(1-c^2\sin^2\beta\,.\,\sin^2\omega).$

This equation is curious, as an *omen* of $\daleth(c\omega)-\dfrac{1}{n}\daleth(h,\theta)$ in the scale whose Index is n; yet it is dearly bought at the price of so much algebra.

PART II.

THE HIGHER THEORY OF F AND E.

CHAPTER VII.

Jacobi's Functions Λ and Θ.

The Algebraic Products S and T.

1. Let q be less than 1, v be real or of the form $\alpha + \beta . \sqrt{-1}$. In fact, we shall presently suppose $v + v^{-1} = 2 \cos x$, or $v = \epsilon^{x\sqrt{-1}}$. Let S and T be functions of q and v, as follows:

$$\left.\begin{matrix} S = (v - v^{-1}) . \left\{\begin{matrix} (1 - q^2v^2)\,(1 - q^4v^2)\,(1 - q^6v^2)\ \ldots \\ (1 - q^2v^{-2})(1 - q^4v^{-2})(1 - q^6v^{-2}) \ldots \end{matrix}\right. \\ T = \left\{\begin{matrix} (1 - qv^2)\,(1 - q^3v^2)\,(1 - q^5v^2)\ \ldots \\ (1 - qv^{-2})(1 - q^3v^{-2})(1 - q^5v^{-2}) \ldots \end{matrix}\right. \end{matrix}\right\} \quad \text{(I call these two the \textit{Pronomi})}$$

which however converge for *all* values of v, because the exponents of q increase beyond all limit.

For S we write $S(v)$ or $S(q, v)$, for T also $T(v)$ or $T(q, v)$.

2. We see at once, that

$$S(v^{-1}) = -S(v) = S(-v),$$

$$T(v^{-1}) = T(v) = T(-v).$$

But further, since, if we write qv in place of v we get

$$S(q, qv) = (qv - q^{-1}v^{-1}) \left\{\begin{matrix} 1 - q^4v^2 \,.\, 1 - q^6v^2 \,.\, 1 - q^8v^2 \ \ldots \\ 1 - v^{-2} . \,1 - q^2v^{-2} . \,1 - q^4v^{-2} \ldots \end{matrix}\right.$$

$$T(q, qv) = \left\{\begin{matrix} 1 - \phantom{q^{-1}}q^3v^2 \,.\, 1 - q^5v^2 \,.\, 1 - q^7v^2 \ \ldots \\ 1 - q^{-1}v^{-2} . \,1 - qv^{-2} . \,1 - q^3v^{-2} \ldots \end{matrix}\right.$$

comparing these with the original series, since

$$(qv - q^{-1}v^{-1})(1 - v^{-2}) = (q^2v^2 - 1)\, q^{-1}v^{-1}(v - v^{-1})\, v^{-1},$$

we obtain (understanding the constant element q in all)

$$S(v) = -qv^2 . S(qv); \quad T(v) = -qv^2 . T(qv). \qquad \text{(a)}.$$

Again, writing $\sqrt{q^{\pm 1}} . v$ in place of v, we similarly find

$$T(v) = -\sqrt{q}v . S(\sqrt{q}v); \quad S(v) = -v^{-1} . T(\sqrt{q^{-1}}v) = v . T(\sqrt{q}v). \qquad \text{(b)}.$$

3. Hence generally,

$$S(mv) = -qm^2v^2S(mqv),$$
$$T(nv) = -qn^2v^2T(nqv).$$

And if $f(v)$ *mean* $\dfrac{S \text{ or } T(mv)}{S \text{ or } T(nv)}$,

$$f(v) = \frac{m^2}{n^2} . f(qv).$$

Or if $f(v)$ *mean* $S(mv) . S(nv)$; to fix ideas (c),

$$\therefore \; f(v) = q^2m^2n^2v^4 . f(qv),$$

and $T(mv) . T(nv) = qmnv^2 . f(\sqrt{q}v)$.

4. Algebraic multiplication of the factors, joined to

$$S(v^{-1}) = S(-v) = -S(v),$$

shows that we may assume

$$S(q, v) = Q(v - v^{-1}) + Q_2(v^3 - v^{-3}) + Q_3(v^5 - v^{-5}) + \&c. \ldots$$

where $Q, Q_2, Q_3 \ldots$ are *determinate* functions of q. Change v to qv.

$$S(q, qv) = Q(qv - q^{-1}v^{-1}) + Q_2(q^3v^3 - q^{-3}v^{-3}) + Q_3(q^5v^5 - q^{-5}v^{-5}) + \ldots$$

Substitute these series in the equation

$$S(q, v) = -qv^2 . S(q, qv)$$

of Art. 2, and compare the coefficients of like powers of v, since we know the coefficients to be determinate. This gives,

$$Q_2 = -q^2Q: \; Q_3 = -q^4Q_2: \; Q_4 = -q^6Q_3: \ldots \; Q_{n+1} = -q^{2n}Q_n:$$

whence $Q_3 = q^{2+4}Q$: $Q_4 = -q^{2+4+6}Q$: ... $Q_{n+1} = (-1)^n q^{2+4+6+\ldots+2n}Q$.

Now $\quad 2 + 4 + 6 + \ldots + 2n = n(n+1)$,

$$\therefore \; S(q, v) = Q\{(v - v^{-1}) - q^{1.2}(v^3 - v^{-3}) + q^{2.3}(v^5 - v^{-5}) - q^{3.4}(v^7 - v^{-7}) + \&c. \ldots\}, \qquad \text{(d)}$$

leaving only Q unknown.

5. Write for the last with n for 1, 2, 3, 4... under Σ,

$$S(v) = Q.\Sigma(-1)^{n-1} q^{(n-1)n} (v^{2n-1} - v^{-2n+1}),$$

and apply the formula (b) of Art. 2,

$$T(v) = -\sqrt{qv}.S(\sqrt{qv}),$$

or
$$T(v) = -\sqrt{qv}.Q.\Sigma(-1)^{n-1}.q^{(n-1)n}(q^{n-\frac{1}{2}}v^{2n-1} - q^{-n+\frac{1}{2}}v^{-2n+1})$$
$$= +Q.\Sigma(-1)^n.q^{(n-1)n}\{q^n v^{2n} - q^{-n+1}v^{-2n+2}\}$$
$$= +Q\Sigma(-1)^n\{q^{nn}v^{2n} - q^{(n-1)^2}v^{-2(n-1)}\}.$$

Restore for $n = 1, 2, 3, \ldots;$

that is
$$T(q, v) = Q\{1 - q^{1\,1}(v^2 + v^{-2}) + q^{2\,2}(v^4 + v^{-4}) - q^{3\,3}(v^6 + v^{-6}) + \&c.\} \quad \text{(e)}.$$

6. To find Q, we may make $v = 1$ in the last, and in the original definition of $T(q, v)$; then

$$Q = \frac{(1-q)^2.(1-q^3)^2.(1-q^5)^2\ldots}{1 - 2q^{1\,1} + 2q^{2\,2} - 2q^{4\,4} + \ldots}.$$

But we shall presently show that

$$Q^{-1} = (1-q^2)(1-q^4)(1-q^6)(1-q^8)\ldots \quad \text{(Art. 15)}.$$

Meanwhile, it is convenient to denote Q by $\phi(q)$, function of q.

7. NOTE. The process in Art. 4 may seem to assume the Postulate, that if a double series

$$A + \frac{A_1 v}{B_1 v^{-1}} + \frac{A_2 v^2}{B_2 v^{-2}} + \frac{A_3 v^3}{B_3 v^{-3}} + \ldots$$

converges, and is equal to a second series of the same form for *all* values of v_1—the coefficients being independent of v_1—then the two series are identical term by term.

But this does not state the case truly, for *it overlooks the enormous increase of the exponent* (with the increase of n) *in* $q^{n\,n}$ *and* q^{nn+1}, while q is < 1. The only needful assumption is that the original S and T in factors (which through the lessening of q^m as m increases, certainly converges) cannot have *more than one* equivalent of the type

$$V_0 + V_1 q^{1\,2} + V_2 q^{2\,3} + V_3 q^{3\,4} + \alpha,$$

and this is really established in Algebra.

The Trigonometrical Series Λ *and* Θ.

8. When we put $v = \epsilon^{x\sqrt{-1}}$ and reduce to functions of x, it is convenient to assume

$$\Lambda(q, x) = 2q^{\frac{1}{4}}\{\sin x - q^{1\cdot 2}\sin 3x + q^{2\cdot 3}\sin 5x - \&c.\}$$
$$\Theta(q, x) = 1 - 2q^{1\cdot 1}\cos 2x + 2q^{2\cdot 2}\cos 4x - 2q^{3\cdot 3}\cos 6x + \&c. \ldots$$

which yield

$$\left.\begin{array}{r}q^{\frac{1}{4}}S(v) = \sqrt{-1}\,.\,\phi(q)\,.\,\Lambda(x)\\ T(v) = \phi(q)\,.\,\Theta(x)\end{array}\right\}\text{(f).}\qquad \text{I call } \Lambda \text{ and } \Theta \text{ the } \textit{Synnomi.}$$

The multiplier $q^{\frac{1}{4}}$ is added to Λ, to make the exponent $\frac{1}{4} + (n-1)\,.\,n$ an algebraic square, as in the other series. This multiplier makes Λ a function of $q^{\frac{1}{4}}$ rather than of q; which is sometimes of importance. For to change $\sqrt{q}$ into $-\sqrt{q}$ changes $q^{\frac{1}{4}}$ into $q^{\frac{1}{4}}\sqrt{-1}$ and hereby Λ into $\sqrt{-1}\Lambda$, while it leaves Θ unchanged.

Also
$$\left.\begin{array}{r}\Lambda(-q, x) = \sqrt[4]{-1}\,.\,\Lambda(q, x):\\ \Theta(-q, x) = \Theta(q, x + \frac{1}{2}\pi).\\ q^{\frac{1}{4}}S(v\sqrt{-1}) = \sqrt{-1}\,.\,\phi(q)\,.\,\Lambda(x + \frac{1}{2}\pi):\\ T(v\sqrt{-1}) = \phi(q)\,.\,\Theta(x + \frac{1}{2}\pi).\end{array}\right\}$$

9. Observe that

$$\left\{\begin{array}{lll}\Lambda(-x) = -\Lambda(x); & \Lambda(x \pm \pi) = -\Lambda(x); & \Lambda(\pi - x) = \Lambda(x).\\ \Theta(-x) = \Theta(x); & \Theta(x \pm \pi) = \Theta(x); & \Theta(\pi - x) = \Theta(x).\end{array}\right\}$$

By Λ^0 we shall denote $\Lambda(\frac{1}{2}\pi - x)$ or its equal $\Lambda(\frac{1}{2}\pi + x)$, and by Θ^0 similarly $\Theta(\frac{1}{2}\pi - x)$ or its equal $\Theta(\frac{1}{2}\pi + x)$; and we shall call $\Lambda^0\Theta^0$ *conjugates* to $\Lambda\Theta$ for a reason which will soon appear.

Also let $\Lambda'(0)$ stand for $\dfrac{\Lambda(x)}{x}$, when $x = 0$, then we have

$$\left.\begin{array}{l}\left.\begin{array}{l}\Lambda'(0) = 2q^{\frac{1}{4}}\{1 - 3q^{1\cdot 2} + 5q^{2\cdot 3} - 7q^{3\cdot 4} + \&c. \ldots\\ \Theta(0) = \qquad 1 - 2q^{1\cdot 1} + 2q^{2\cdot 2} - 2q^{3\cdot 3} + \&c. \ldots\end{array}\right\}\\ \left.\begin{array}{l}\Lambda(\frac{1}{2}\pi) = 2q^{\frac{1}{4}}\{1 + q^{1\cdot 2} + q^{2\cdot 3} + q^{3\cdot 4} + \&c. \ldots\\ \Theta(\frac{1}{2}\pi) = \qquad 1 + 2q^{1\cdot 1} + 2q^{2\cdot 2} + 2q^{3\cdot 3} + \&c. \ldots\end{array}\right\}\end{array}\right\}.$$

10. Writing $\epsilon^{x\sqrt{-1}}$ for v in the primitive form of S and T, we get from (f),

$$\phi(q) . \Lambda(q, x)$$
$$= 2q^{\frac{1}{4}} \sin x (1 - 2q^2 \cos 2x + q^4)(1 - 2q^4 \cos 2x + q^8)(1 - 2q^6 \cos 2x + q^{12})\ldots$$

and $\phi(q)\Theta(q, x)$

$$= (1 - 2q \cos 2x + q^2)(1 - 2q^3 \cos 2x + q^6)(1 - 2q^5 \cos 2x + q^{10}) \ldots \text{(g)},$$

of which we have as particular cases

$$\left.\begin{aligned} \phi(q) . \Lambda'(0) &= 2q^{\frac{1}{4}}(1-q^2)^2(1-q^4)^2(1-q^6)^2 \ldots \\ \phi(q) . \Theta(0) &= (1-q)^2(1-q^3)^2(1-q^5)^2 \ldots \\ \phi(q) . \Lambda(\tfrac{1}{2}\pi) &= 2q^{\frac{1}{4}}(1+q^2)^2(1+q^4)^2(1+q^6)^2 \ldots \\ \phi(q) . \Theta(\tfrac{1}{2}\pi) &= (1+q)^2(1+q^3)^2(1+q^5)^2 \ldots \end{aligned}\right\} .$$

These, compared with the four equations of the last Art. give four values of $\phi(q)$, one of which was obtained in Art. 6.

11. For a moment, write

$$\left.\begin{aligned} \alpha &= 1 - q . 1 - q^3 . 1 - q^5 \ldots \\ \beta &= 1 + q . 1 + q^3 . 1 + q^5 \ldots \end{aligned}\right\}, \qquad \left.\begin{aligned} \gamma &= 1 - q^2 . 1 - q^4 . 1 - q^6 \ldots \\ \delta &= 1 + q^2 . 1 + q^4 . 1 + q^6 \ldots \end{aligned}\right\}$$

which give

$$\left.\begin{aligned} \phi(q) . \Lambda'(0) &= 2q^{\frac{1}{4}}\gamma^2 ; \qquad & \phi(q)\Lambda(\tfrac{1}{2}\pi) &= 2q^{\frac{1}{4}}\delta^2 ; \\ \phi(q) . \Theta(0) &= \alpha^2 ; & \phi(q)\Theta(\tfrac{1}{2}\pi) &= \beta^2 \end{aligned}\right\} .$$

Also

$$\alpha\beta = 1 - q^2 . 1 - q^6 . 1 - q^{10} \ldots ; \quad \gamma\delta = 1 - q^4 . 1 - q^8 . 1 - q^{12} \ldots ,$$

so that $\alpha\beta . \gamma\delta = 1 - q^2 . 1 - q^4 . 1 - q^6 . 1 - q^8 \ldots = \gamma ;$

or $$\alpha\beta\delta = 1 ;$$

$$\therefore \quad \phi^3(q) . \Lambda(\tfrac{1}{2}\pi) . \Theta(0) . \Theta(\tfrac{1}{2}\pi) = 2q^{\frac{1}{4}},$$

or $$\Lambda(\tfrac{1}{2}\pi) . \Theta(0) . \Theta(\tfrac{1}{2}\pi) = \frac{2q^{\frac{1}{4}}}{\phi^3(q)} .$$

12. It is known in Trigonometry, that if $\chi(q, x)$ stand for

$$1 - 2q \cos 2x + q^2,$$

$$\therefore \quad \chi(q^n, nx) = \chi(q, x) . \chi\left(q, x + \frac{\pi}{n}\right) \ldots \chi\left(q, x + \frac{n-1}{n}\pi\right) ;$$

also

$$2(q^n)^{\frac{1}{4}} \sin nx = 2q^{\frac{1}{4}} \sin x . 2q^{\frac{1}{4}} \sin\left(x + \frac{\pi}{n}\right) \ldots 2q^{\frac{1}{4}} \sin\left(x + \frac{n-1}{n.}\pi\right) ;$$

and every factor of Λ and Θ in Art. 10 is included in these forms.

Resolving hereby every factor of $\Lambda(q^n, nx)$ and of $\Theta(q^n, nx)$ into n factors, and recomposing, if $\Lambda_r \Theta_r$ stand for

$$\Lambda\left(q,\ x+\frac{r}{n}\pi\right), \qquad \Theta\left(q,\ x+\frac{r}{n}\pi\right),$$

you get

$$\left.\begin{aligned}\phi(q^n)\,.\,\Lambda(q^n,\ nx) &= \phi^n(q)\,.\,\Lambda\Lambda_1\Lambda_2\ldots\Lambda_{n-1},\\ \phi(q^n)\,.\,\Theta(q^n,\ nx) &= \phi^n(q)\,.\,\Theta\Theta_1\Theta_2\ldots\Theta_{n-1}\end{aligned}\right\}\ \text{(h)}.$$

In the case of $n=2$, write

$$\psi(q^2) \text{ for } \phi(q^2) \div \phi^2(q);$$

$$\therefore\quad \left.\begin{aligned}\psi(q^2)\,.\,\Lambda(q^2,\ 2x) &= \Lambda\Lambda^0;\\ \psi(q^2)\,.\,\Theta(q^2,\ 2x) &= \Theta\Theta^0\end{aligned}\right\}\ \text{(h}'\text{)};$$

and *we shall presently show* that

$$\psi(q^2) = \Theta(q^2,\ 0).$$

13. Recurring to the original series for Θ, we have at once

$$\Theta = 1 - 2q^{1\cdot 1}\cos 2x + 2q^{2\cdot 2}\cos 4x - 2q^{3\cdot 3}\cos 6x + \&\text{c.}\ldots,$$

and $\quad \Theta^0 = 1 + 2q^{1\cdot 1}\cos 2x + 2q^{2\cdot 2}\cos 4x + 2q^{3\cdot 3}\cos 6x + \&\text{c.}\ldots;$

whence

$$\Theta^0 + \Theta = 2\{1 + 2q^{2\cdot 2}\cos 4x + 2q^{4\cdot 4}\cos 8x + \&\text{c.}\ldots\} = 2\Theta(q^4,\ 2x+\tfrac{1}{2}\pi),$$

$$\begin{aligned}\Theta^0 - \Theta &= 4\{q^{1\cdot 1}\cos 2x + q^{3\cdot 3}\cos 6x + q^{5\cdot 5}\cos 10x + \&\text{c.}\}\\ &= 2\Lambda(q^4,\ 2x+\tfrac{1}{2}\pi).\end{aligned}$$

Change q to $\sqrt{q}$, and let Θ', Λ' stand for

$$\Theta(q^2,\ 2x+\tfrac{1}{2}\pi), \qquad \Lambda(q^2,\ 2x+\tfrac{1}{2}\pi);$$

$$\therefore\quad \Theta^0(\sqrt{q},\ x) = \Theta' + \Lambda', \quad \Theta(\sqrt{q},\ x) = \Theta' - \Lambda';$$

whence $\qquad \Theta'^2 - \Lambda'^2 = \Theta(\sqrt{q},\ x)\,.\,\Theta^0(\sqrt{q},\ x)$

which further $\qquad = \psi(q)\,.\,\Theta(q,\ 2x),$

by last article (h′), or, changing $2x$ to x, and then x to $x \pm \frac{1}{2}\pi$,

$$\Theta^2(q^2,\ x) - \Lambda^2(q^2,\ x) = \psi(q)\,.\,\Theta^0(q,\ x).$$

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14. *Values of certain Constants.*

In the last, let $x=0$, therefore

$$\Theta^2(q^2,\ 0) = \psi(q)\,.\,\Theta(q,\ \tfrac{1}{2}\pi),$$

and making $x=0$ in the last equation of Art. 12,

$$\psi(q^2)\,\Theta(q^2,\ 0) = \Theta(q,\ 0)\,.\,\Theta(q,\ \tfrac{1}{2}\pi).$$

Eliminate $$\Theta(q, \tfrac{1}{2}\pi);$$

$$\therefore \quad \frac{\psi(q)}{\Theta(q, 0)} = \frac{\Theta(q^2, 0)}{\psi(q^2)}.$$

In this, change q into $q^2, q^4, q^8, q^{16} \ldots,$

and the equation is continued into

$$= \frac{\psi(q^4)}{\Theta(q^4, 0)} = \frac{\Theta(q^8, 0)}{\psi(q^8)} = \frac{\psi(q^{16})}{\Theta(q^{16}, 0)} = \text{\&c.} \ldots.$$

Now when $n = \infty$,

$$\Theta(q^n, 0) = \Theta(0, 0) = 1.$$

Also $$\phi(q^n) = \phi(0) = 1,$$

and consequently $$\psi(q^n) = 1,$$

when $n = \infty$. Hence the ratio being finally $= 1 : 1$, we have absolutely

$$\psi(q) = \Theta(q, 0)$$

which fulfils the promise at the close of Art. 12.

15. We can now show that

$$\frac{1}{\phi(q)} = 1 - q^2 . 1 - q^4 . 1 - q^6 . 1 - q^8 \ldots \quad \text{(i)}.$$

For since

$$\frac{\phi^2(q)}{\phi^2(\sqrt{q})} = \phi(q) . \psi(q) = \phi(q) . \Theta(q, 0) = \alpha^2, \text{ of Art. 11}:$$

let us for a moment call $\alpha = f(q)$, as a function of q; therefore

$$\frac{\phi(q)}{\phi(\sqrt{q})} = f(q).$$

For q write $$q^2, q^4, q^8, \ldots, q^{2^n};$$

multiply all the results together; therefore

$$\frac{\phi(q^m)}{\phi(\sqrt{q})} = f(q) . f(q^2) . f(q^4) \ldots f(q^m), \text{ if } m = 2^n.$$

Now in $f(q)$ or $(1 - q)(1 - q^3)(1 - q^5) \ldots$ the exponents of q are the series of odd numbers. In $f(q^2)$ they are $2 \times$ the odd numbers. In $f(q^4)$ they are $2^2 \times$ the odd numbers; and so on. Hence, if in the product $f(q) . f(q^2) \ldots$ we take $n = \infty$, the combination of all the factors exhausts the series of natural numbers in the exponents.

Also

$$q^m = 0,\ \phi(q^m) = 1;$$

or

$$\frac{1}{\phi(\sqrt{q})} = fq \,.\, f(q^2) \,.\, f(q^4) \,.\, f(q^8) \ldots \textit{ad infin.}$$
$$= (1-q)(1-q^2)(1-q^3)(1-q^4) \ldots \&c.$$

Change $\sqrt{q}$ to q, and you get the equation which opens this article.

Cor. Thus $\qquad \phi(q) = \gamma^{-1}$ of Art. 11;

and

$$\Lambda'(0) = 2q^{\frac{1}{4}}\gamma^3;$$

or

$$\Lambda'(0) = \Lambda(\tfrac{1}{2}\pi) \,.\, \Theta(0) \,.\, \Theta(\tfrac{1}{2}\pi) \quad \text{(j)}.$$

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16. Since in 1st equation of Art. 10, by changing q to $\sqrt{q}$,

$$\phi(\sqrt{q})\,\Lambda(\sqrt{q},\ x)$$
$$= 2q^{\frac{1}{8}}\sin x\,(1 - 2q\cos 2x + q^2)(1 - 2q^2\cos 2x + q^4) \ldots,$$

it is clear by combining two first equations of Art. 10 that

$$q^{\frac{1}{8}}\phi(\sqrt{q})\,\Lambda(\sqrt{q},\ x) = \phi^2(q) \,.\, \Lambda(q,\ x) \,.\, \Theta(q,\ x).$$

Multiply by $\qquad \Lambda^0(\sqrt{q},\ x)$,

therefore by (h′) of Art. 12,

$$P \,.\, \Theta(q,\ 0)\,\Lambda(q,\ 2x) = \Lambda(q,\ x)\,\Theta(q,\ x)\,\Lambda^0(\sqrt{q},\ x);$$

if for a moment P stands for

$$q^{\frac{1}{8}}\phi(\sqrt{q}) \div \phi^2(q).$$

Now when x is evanescent,

$$\frac{\Lambda(q,\ 2x)}{\Lambda(q,\ x)} = 2;$$

$$\therefore\ 2P = \Lambda(\sqrt{q},\ \tfrac{1}{2}\pi).$$

Also by h', with $\sqrt{q}$ for q,

$$\Theta(q,\ 0),\ \Lambda(q,\ 2x)$$
$$= \Lambda(\sqrt{q},\ x)\ \Lambda^0 \,.\, (\sqrt{q},\ x);$$

whence $\qquad \Lambda(\sqrt{q},\ \tfrac{1}{2}\pi) \,.\, \Lambda(\sqrt{q},\ x) = 2\Lambda(q,\ x) \,.\, \Theta(q,\ x) \quad$ (k).

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17. *Other Sums or Products of* Λ *and* Θ *simplified.*

In the last equation of Art. 13 write $\sqrt{q}$ for q;

$$\therefore\ \Theta^2(q,\ x) - \Lambda^2(q,\ x) = \Theta(\sqrt{q},\ 0)\,\Theta^0(\sqrt{q},\ x).$$

Change $\sqrt{q}$ to $-\sqrt{q}$, which changes each Θ on the right to its conjugate, and Λ into $\sqrt{-1} \,.\, \Lambda$, without altering $\Theta(q,\ x)$;

$$\therefore\ \Theta^2(q,\ x) + \Lambda^2(q,\ x) = \Theta(\sqrt{q},\ \tfrac{1}{2}\pi) \,.\, \Theta(\sqrt{q},\ x).$$

But by Art. 13,

$$\Theta(\sqrt{q},\ x) = \Theta' - \Lambda', \quad \Theta^0(\sqrt{q},\ x) = \Theta' + \Lambda.$$

Let $\theta\lambda$ be the values of $\Theta'\Lambda'$ when $x=0$; or

$$\theta = \Theta(q^2,\ \tfrac{1}{2}\pi), \quad \lambda = \Lambda(q^2,\ \tfrac{1}{2}\pi);$$

then $\quad \Theta^2 - \Lambda^2 = (\theta-\lambda)(\Theta' + \Lambda'); \quad \Theta^2 + \Lambda^2 = (\theta+\lambda)(\Theta' - \Lambda');$

$$\left.\begin{array}{ll} \text{whence} & \Theta^2 = \theta\Theta' - \lambda\Lambda'; \quad \Lambda^2 = \lambda\Theta' - \theta\Lambda': \\ \text{change } x \text{ to } x+\tfrac{1}{2}\pi; & \\ & \Theta^{02} = \theta\Theta' + \lambda\Lambda'; \quad \Lambda^{02} = \lambda\Theta' + \theta\Lambda'. \end{array}\right\} \quad (1).$$

18. Combining $\Theta^2 + \Lambda^2 = \Theta(\sqrt{q},\ \tfrac{1}{2}\pi)\,\Theta(\sqrt{q},\ x)$

with $$2\Theta\Lambda = \Lambda(\sqrt{q},\ \tfrac{1}{2}\pi)\,\Lambda(\sqrt{q},\ x),$$

we get $\;(\Theta \pm \Lambda)^2 = \Theta(\sqrt{q},\ \tfrac{1}{2}\pi)\,\Theta(\sqrt{q},\ x) \pm \Lambda(\sqrt{q},\ \tfrac{1}{2}\pi)\,\Lambda(\sqrt{q},\ x).$

Again,
$$\begin{aligned} \Theta^4 - \Lambda^4 &= (\Theta^2 + \Lambda^2)(\Theta^2 - \Lambda^2) \\ &= \Theta(\sqrt{q},\ 0)\,\Theta(\sqrt{q},\ \tfrac{1}{2}\pi)\,.\,\Theta(\sqrt{q},\ x)\,.\,\Theta^0(\sqrt{q},\ x) \\ &= \Theta(q,\ 0)\,.\,\Theta(q,\ \pi)\,.\,\Theta(q,\ 0)\,.\,\Theta(q,\ 2x) \\ &= \Theta^3(q,\ 0)\,.\,\Theta(q,\ 2x). \end{aligned}$$

Just as $(\cos x)^n(\sin x)^n$ are reduced to linear sines or cosines of multiples of x, so here we see the powers of Θ and Λ reduced.

In the last, change q to $-q$, Λ to $\sqrt[4]{-1}\,.\,\Lambda$;

$$\therefore \quad \Theta^{04} + \Lambda^4 = \Theta^3(q,\ \tfrac{1}{2}\pi)\,\Theta^0(q,\ 2x).$$

Again, in Art. 13,

$$\begin{aligned} \Theta^{02} - \Theta^2 &= 2\Lambda(q^4,\ 2x+\tfrac{1}{2}\pi)\,.\,2\Theta(q^4,\ 2x+\tfrac{1}{2}\pi) \\ &= 2\Lambda(q^2,\ \tfrac{1}{2}\pi)\,.\,\Lambda(q^2,\ 2x+\tfrac{1}{2}\pi), \text{ by Art. 16.} \end{aligned}$$

The number of such formulas seems inexhaustible.

Problem Cardinal for Elliptics.

19. To develop $X = \Lambda^0 \dfrac{d\Lambda}{dx} - \Lambda \dfrac{d\Lambda^0}{dx}$, in linear sines or cosines.

Revert to the Pronomi.

Assume the function χ such that

$$\chi(v)\,d\log v = S(v\sqrt{-1})\,dS(v) - S(v)\,dS(v\sqrt{-1}),$$

which means $$\chi(v) = \sqrt{-1}\,.\,\phi^2(q)\,.\,X.$$

Also by Art. 3,

$$\frac{S(v)}{S(v\sqrt{-1})} = \frac{-S(qv)}{S(qv\sqrt{-1})}. \quad \text{Differentiate,}$$

$$\therefore \quad \frac{\chi(v)}{S^2(v\sqrt{-1})} = \frac{-\chi(qv)}{S^2(qv\sqrt{-1})},$$

whence again by (a) in Art. 2,

$$\chi(v) = -q^2v^4 \cdot \chi(qv).$$

Observing that

$$\frac{dS(v)}{d\log v} = \phi(q) \cdot \{(v+v^{-1}) - 3q^{1\cdot 2}(v^3+v^{-3}) + 5q^{2\cdot 3}(v^5+v^{-5}) - \&c. \dots\},$$

we see that $\qquad S(v\sqrt{-1}) \cdot \dfrac{dS(v)}{d\log v}$

has the form $\qquad N + N_1(v^2+v^{-2}) + N_2(v^4+v^{-4}) + \&c.$

which gives

$$S(-v) \cdot \frac{dS(v\sqrt{-1})}{d\log(v\sqrt{-1})} = N - N_1(v^2+v^{-2}) + N_2(v^4+v^{-4}) - \&c. \dots$$

of which the sum is

$$\chi(v) = 2N + 2N_2(v^4+v^{-4}) + 2N_4(v^8+v^{-8}) + \&c. \dots$$

To this apply the equation

$$\chi(v) = -q^2v^4 \cdot \chi(qv),$$

and you get

$$N_2 = -Nq^{2\cdot 1}; \quad N_4 = -N_2 \cdot q^{2\cdot 3}; \dots \quad N_{2n} = -N_{2n-2}q^{2(2n-1)};$$

whence $\qquad N_{2n} = (-1)^n N q^{2(1+3+5+\dots+2n-1)} = (-1)^n N q^{2nn};$

or $\chi(v) = 2N\{1 - q^2(v^4+v^{-4}) + q^{2(2.2)}(v^8+v^{-8}) - q^{2(3.3)}(v^{12}+v^{-12}) + \dots\}.$

Hence, neglecting the constant multipliers, when x varies and q is constant, X varies as $\Theta(q^2, 2x)$ which is in linear cosines. But it is more convenient to replace the last by $\Theta\Theta^0$. Let $X = m\Theta\Theta^0$.

20. To find m, let x be infinitesimal;

$$\frac{d\Lambda}{dx} = \Lambda'(0);$$

then $\qquad \Lambda(\tfrac{1}{2}\pi) \cdot \Lambda'(0) = m \cdot \Theta(0) \cdot \Theta(\tfrac{1}{2}\pi).$

But by (j) of 15, $\Lambda'(0) = \Lambda(\tfrac{1}{2}\pi) \cdot \Theta(0) \cdot \Theta(\tfrac{1}{2}\pi);$

$$\therefore \quad \Lambda^2(\tfrac{1}{2}\pi) = m.$$

Or $\qquad \Lambda^0 \dfrac{d\Lambda}{dx} - \Lambda\dfrac{d\Lambda^0}{dx} = \Lambda^2(\tfrac{1}{2}\pi) \cdot \Theta \cdot \Theta^0 \quad \text{(m).}$

COR. [Since $\chi(v)$ *varies as* $T(q^2, v^2)$, change v to $v\sqrt{q}$, and, by algebraic routine, you get

$$\Theta^0 \frac{d\Theta}{dx} - \Theta \frac{d\Theta^0}{dx} = \Lambda^2(\tfrac{1}{2}\pi) \,.\, \Lambda \,.\, \Lambda^0 \quad (\text{m}').]$$

Legendre's Elliptic Trigonometry.

21. The product of any two Synnomi may now be made linear, as Θx, Θy. Begin from the Pronomi, making $f(v) = T(mv)\,T(nv)$, then by reasoning as in Art. 3 above

$$f(v) = q^2 m^2 n^2 v^4 f(qv).$$

From Art. 5,

$$T(mv) = Q\,\{1 - q^{1\,1}(m^2v^2 + m^{-2}v^{-2}) + q^{2.2}(m^4v^4 + m^{-4}v^{-4}) - q^{3\,3} \ldots \&\text{c.}\}$$
$$T(nv) = Q\,\{1 - q^{1\,1}(n^2v^2 + n^{-2}v^{-2}) + q^{2\,2}(n^4v^4 + n^{-4}v^{-4}) - q^{3\,3} \ldots \&\text{c.}\}.$$

By the process of multiplying these, we get, in form,

$$f(v) = R - (R_1v^2 + p_1v^{-2}) + R_2v^4 + p_2v^{-4}) - (R_3v^6 + p_3v^{-6}) + \&\text{c.} \quad (\text{n}),$$

in which no R and no p involves v. The first term R is the sum of the terms in which the exponent of v is zero. That is, if for a moment we write

$$T(q,\ v) = Q - Q'(v^2 + v^{-2}) + Q''(v^4 + v^{-4}) - Q''' \,.\, \&\text{c.};$$

then $\quad R = Q^2 + Q'^2(m^2n^{-2} + m^{-2}n^2) + Q''^2(m^4n^{-4} + m^{-4}n^4) + \&\text{c.},$

so that $\quad \dfrac{R}{Q^2} = 1 + q^{2(1\ 1)}\left(\dfrac{m^2}{n^2} + \dfrac{n^2}{m^2}\right) + q^{2(2\ 2)}\left(\dfrac{m^4}{n^4} + \dfrac{n^4}{m^4}\right) + \&\text{c.},$

or in shorter notation

$$\frac{R}{Q^2} = \frac{T(q^2,\ mn^{-1}\sqrt{-1})}{\phi(q^2)}.$$

The term R_1 is the collection of terms in which the exponent of v is 2, or $\quad R_1 = QQ'(m^2 + n^2) + Q'Q''(m^4n^{-2} + n^4m^{-2})$

$$+ Q''Q'''(m^6n^{-4} + n^{-6}m^{-4}) + \&\text{c.},$$

or $$\frac{R_1}{Q^2} = \frac{qmn}{\phi(q^2)} \cdot \frac{S(q^2,\ mn^{-1}\sqrt{-1})}{\sqrt{-1}}.$$

Neither R nor $\dfrac{R_1}{mn}$ is altered, if m, n be changed to m^{-1}, n^{-1}: hence the change alters R_1 into $R_1m^{-2}n^{-2}$. Also $T(mv)\,T(nv)$ remains unaltered if m, n, v become m^{-1}, n^{-1}, v^{-1}; yet this exchanges every R

with its companion p_r. Hence when R_r is known, we deduce p_r from it by changing m, n into m^{-1}, n^{-1}.

22. Apply the test $f(v) = m^2n^2q^2v^4f(qv)$ to the series (n), and compare the results, first when R has an even subscript. Then

$$R_2 = q^2m^2n^2R;\ \ R_4 = q^6m^2n^2R_2;\ \dots\ R_{2r} = q^{4r-2}m^2n^2R_{2r-2}.$$

We deduce $\quad R_{2n} = q^{2+6+10+\dots+(4r-2)}m^{2r}n^{2r}R = q^{2rr}m^{2r}n^{2r}R.$

Next, for R with an odd subscript,

$$R_3 = q^4m^2n^2R_1;\ \ R_5 = q^8m^2n^2R_3;\ \dots\ R_{2r+1} = q^{4r}m^2n^2R_{2r-1};$$

which gives $\quad R_{2r+1} = q^{2r(r+1)}\, m^2n^2R_1,$

whence further $\quad p_{2r} = q^{2rr} . (mn)^{-2r}R;$

$$p_{2r+1} = q^{2r(r+1)} . (mn)^{-2r-2}R_1.$$

To recompose let $\quad f(v) = RM + R_1N.$

Then $M = 1 + q^{2(1\ 1)}(m^2n^2v^4 + m^{-2}n^{-2}v^{-4}) + q^{2(2.2)}(m^4n^4v^8 + m^{-4}n^{-4}v^{-8}) + \&c.,$

or $\quad \phi(q^2) . M = T(q^2,\ mnv^2\sqrt{-1}).$

Next $\quad mnN = (mnv^2 + m^{-1}n^{-1}v^{-2}) + q^{2(1\ 2)}(m^3n^3v^6 + m^{-3}n^{-3}v^{-6}) + \&c.,$

or $\quad \sqrt{-1} . \phi(q^2)\, mnN = S(q^2,\ mnv^2\sqrt{-1}).$

We have now identically

$$T(mv)\, T(nv) = RM + R_1N \quad \text{(o)}.$$

If now we simplify by assuming $v = 1$, it does not thereby lose generality, m and n being arbitrary. Assume

$$m = \epsilon^{x\sqrt{-1}},\ \ n = \epsilon^{y\sqrt{-1}};$$

$$\therefore\ \ T(m) = Q . \Theta(x),\ \ T(n) = Q . \Theta(y),$$

the basis q being understood. Also since by Art. 8, $T(v\sqrt{-1})$

$$[\text{when } v = \epsilon^{x\sqrt{-1}}] = \phi(q) . \Theta^0(x),$$

here $\quad T(q^2,\ mn^{-1}\sqrt{-1}) = \phi(q^2) . \Theta^0(x-y)$

since mn^{-1} now $= \epsilon^{(x-y)\sqrt{-1}}$, and

$$S(q^2,\ mn^{-1}\sqrt{-1}) = q^{-\frac{1}{2}} . \sqrt{-1} . \phi(q^2)\, \Lambda^0(x-y).$$

Further $\quad T(q^2,\ mn\sqrt{-1}) = \phi(q^2) . \Theta^0(x+y);$

$$S(q^2,\ mn\sqrt{-1}) = \phi(q^2) . \Lambda^0(x+y).$$

Now, attending to $\sqrt{-1} . \sqrt{-1}$ in denominator of R_1N, equation (o) changes into $\Theta(q,\ x) . \Theta(q,\ y) = \Theta^0(q^2,\ x-y) . \Theta^0(q^2,\ x+y)$

$$- \Lambda^0(q^2,\ x-y) . \Lambda^0(q^2,\ x+y).$$

23. Before making $v=1$, we may change v to $\sqrt{qv}$, and transform by equation (b) of Art. 2. Then passing from Pronomi to Synnomi, by mere algebraic routine we obtain

$$\Lambda(q,\ x)\,.\,\Lambda(q,\ y)=\Theta^0(q^2,\ x+y)\,\Lambda^0(q^2,\ x-y)$$
$$-\,\Theta^0(q^2,\ x-y)\,\Lambda^0(q^2.\ x+y).$$

To try our algebraic details; *first*, let $y=0$; then the result in Art. 21 is true by 1st equation in 17, and that in Art. 22 becomes

$$\Theta x\,.\,\Theta(0)=-\,\Lambda^{02}(q^2,\ x)+\Theta^{02}(q^2,\ x);$$

which merges in $\Theta(\sqrt{q},\ x)\,.\,\Theta(\sqrt{q},\ 0)=\Theta^{02}-\Lambda^{02}$,

if q^2 is changed to $\sqrt{q}$; and agrees with Art. 17. *Next*, let $x=y$;

$$\therefore\quad -\,\Lambda^2(x)=-\,\Lambda(q^2,\ \tfrac{1}{2}\pi)\,\Theta^0(q^2,\ 2x)+\Theta(q^2,\ \tfrac{1}{2}\pi)\,.\,\Lambda^0(q^2,\ 2x),$$

which is nothing but $\Lambda^2=\lambda\Theta'-\theta\Lambda'$ of Art. 17.

24. Change y to $y+\frac{1}{2}\pi$, and pay attention to the signs;

$$\therefore\quad \Lambda x\Lambda^0 y=\Theta(q^2,\ x+y)\,\Lambda(q^2,\ x-y)+\Lambda(q^2,\ x+y)\,.\,(q^2,\ x-y)$$
$$\Theta x\Theta^0 y=\Lambda(q^2,\ x+y)\,\Lambda(q^2,\ x-y)+\Theta(q^2,\ x+y)\,.\,(q^2,\ x-y).$$

In the last pair we may exchange x and y, which produces two new equations: then by addition and subtraction we elicit four equations as follows:

$$\left.\begin{aligned}\Theta x\Theta^0 y+\Theta y\Theta^0 x&=2\Theta(q^2,\ x+y)\,.\,\Theta(q^2,\ x-y)\\ \Theta x\Theta^0 y-\Theta y\Theta^0 x&=2\Lambda(q^2,\ x+y)\,.\,\Lambda(q^2,\ x-y)\\ \Lambda x\Lambda^0 y+\Lambda y\Lambda^0 x&=2\Lambda(q^2,\ x+y)\,.\,\Theta(q^2,\ x-y)\\ \Lambda x\Lambda^0 y-\Lambda y\Lambda^0 x&=2\Lambda(q^2,\ x-y)\,.\,\Theta(q^2,\ x+y)\end{aligned}\right\}\quad(\text{p}).$$

Beautiful as are these equations they are not needed in what here follows, except to show that they contain, and might supersede, our grand theorems in Chapters IV. and V. above.

Transition to Elliptics.

25. In Art. 17 we found $\Theta^2\Lambda^2\Theta^{02}\Lambda^{02}$ to be each linear in Θ' and Λ', when x alone is variable. We count θ and λ constant, as functions of q only, for we had

$$\theta=\Theta(q^2,\ \tfrac{1}{2}\pi),\quad \lambda=\Lambda(q^2,\ \tfrac{1}{2}\pi).$$

Thus we call the right-hand member *linear* in

$$\Theta^2=\theta\Theta'-\lambda\Lambda',\ \text{and}\ \Lambda^2=\lambda\Theta'-\theta\Lambda'.$$

With b and c as *disposable constants*, form the combination

$$y = \Lambda^2 + b\Lambda^{02} - c\Theta^2,$$

or
$$y = (\lambda\Theta^{\backprime} - \theta\Lambda^{\backprime}) + b\,(\lambda\Theta^{\backprime} + \theta\Lambda^{\backprime}) - c\,(\theta\Theta^{\backprime} - \lambda\Lambda^{\backprime}).$$

If herein b and c be so assumed, as to make the coefficients of $\Theta^{\backprime}$ and of $\Lambda^{\backprime}$ vanish separately, we shall deduce $y = 0$, or

$$\Lambda^2 + b\Lambda^{02} = c\Theta^2,$$

under the two conditions

$$\lambda + b\lambda - c\theta = 0, \quad -\theta + b\theta + c\lambda = 0,$$

that is
$$\frac{1+b}{c} = \frac{\theta}{\lambda},$$

and
$$\frac{1-b}{c} = \frac{\lambda}{\theta}.$$

Multiply these together,

$$\therefore \quad \frac{1-b^2}{c^2} = 1, \qquad b^2 + c^2 = 1.$$

Hence b and c are suited for *moduli* to an elliptic integral.

Next dividing 2nd equation by 1st,

$$\frac{1-b}{1+b} = \frac{\lambda^2}{\theta^2},$$

whence
$$b = \frac{\theta^2 - \lambda^2}{\theta^2 + \lambda^2};$$

and finally
$$c = \frac{2\theta\lambda}{\theta^2 + \lambda^2};$$

which show b and c to be always finite and real.

26. With these values of b and c we always find

$$\begin{array}{ll} & \left.\begin{aligned} &\Lambda^2 + b\Lambda^{02} = c\Theta^2 \\ \therefore\ &\Lambda^{02} + b\Lambda^2 = c\Theta^{02} \end{aligned}\right\} \\ \text{Change } x \text{ to } x + \tfrac{1}{2}\pi; & \\ \text{Eliminate } \Lambda^{02}; & \left.\begin{aligned} &\Theta^2 - b\Theta^{02} = c\Lambda^2 \\ &\Theta^{02} - b\Theta^2 = c\Lambda^{02} \end{aligned}\right\} \\ \text{Again change } x \text{ to } x + \tfrac{1}{2}\pi; & \end{array}$$

Thus any one of these four equations reproduces the other three.

Putting $x = 0$, $\Lambda = 0$ in the 2nd of these, you deduce, first,

$$\sqrt{c} = \frac{\Lambda^0}{\Theta^0},$$

which now means
$$\sqrt{c} = \frac{\Lambda\,(\frac{1}{2}\pi)}{\Theta\,(\frac{1}{2}\pi)},$$

or $$\sqrt{c} = \frac{2q^{\frac{1}{4}}(1 + q^{1.2} + q^{2.3} + q^{3.4} + \ldots)}{1 + 2q^{1.1} + 2q^{2.2} + 2q^{3.3} + \&c.},$$

by the Synnomi. This also shows, that with q evanescent $\sqrt{c} = 2q^{\frac{1}{4}}$, that is, $\frac{1}{4}c$ converges to $\sqrt{q}$.

But next, our assumption $x = 0$, yields also from the 3rd equation

of the four $$\sqrt{b} = \frac{\Theta}{\Theta^0},$$

which here means $$\sqrt{b} = \frac{\Theta(0)}{\Theta(\frac{1}{2}\pi)},$$

or $$\sqrt{b} = \frac{1 - 2q^{1.1} + 2q^{2.2} - 2q^{3.3} + \&c.}{1 + 2q^{1.1} + 2q^{2.2} + 2q^{3.3} + \&c.}.$$

27. Suppose c_1 related to q^2 as c is related to q.

This means, as $$\sqrt{c} = \frac{\Lambda(q, \frac{1}{2}\pi)}{\Theta(q, \frac{1}{2}\pi)},$$

so $$\sqrt{c_1} = \frac{\Lambda(q^2, \frac{1}{2}\pi)}{\Theta(q^2, \frac{1}{2}\pi)};$$

or exactly $$\sqrt{c_1} = \frac{\lambda}{\theta},$$

then $$c_1 = \frac{\lambda^2}{\theta^2}.$$

But by Art. 1 this yields $$c_1 = \frac{1-b}{1+b}.$$

This was the cardinal relation of Landen's scale. If now we take

$$b_1 = \sqrt{(1 - c_1^2)},$$

c_1 and b_1 become related to c and b, exactly as if formed from q^2 by the same law as c and b from q.

Therefore in general, if $c, c_1, c_2, c_3 \ldots c_n$ follow Landen's law, and $q, q_1, q_2, q_3 \ldots q_n$ are so formed from q that each is *the square* of the preceding, c_n is related to q_n exactly as the first c to the first q.

To *invert* the series of c into $c, c', c'', c''' \ldots$ corresponds then with a q series, of which each is the *square root* of that immediately preceding.

To exchange q into $-q$ in the equation

$$\sqrt{b} = \frac{\Theta(q,\ 0)}{\Theta(q,\ \tfrac{1}{2}\pi)}$$

evidently inverts the fraction and changes b into b^{-1}; therefore also changes c into $cb^{-1}\sqrt{-1}$.

Cor. Also if we multiply numerator and denominator of the last fraction by the numerator, the denominator becomes

$$\Theta(q,\ 0) \,.\, \Theta(q,\ \tfrac{1}{2}\pi),$$

which $= \Theta^2(q^2, 0)$ by Art. 12 above, equation (h′). Extracting then the square root

$$\sqrt[4]{b} = \frac{\Theta(q,\ 0)}{\Theta(q^2,\ 0)}.$$

28. Relation of the *basis* q to our *promodulus* ρ.

When q is evanescent, we have seen that

$$\sqrt{c} = 2q^{\frac{1}{4}}.$$

Hence when n is large,

we have approximately $\quad \sqrt{c_n} = 2(q_n)^{\frac{1}{4}},$

or $\quad c_n = 4\sqrt{q_n} \quad \text{or} \quad \tfrac{1}{4}c_n = q_{n-1},$

whence $\quad \log\dfrac{1}{q_{n-1}} = \log\dfrac{4}{c_n}.$

But $\quad q_{n-1} = q_{n-2}{}^2;$

$$\therefore\ \log\frac{1}{q_{n-2}} = 2^{-1}\,.\,\log\frac{4}{c_n}.$$

Again $\quad q_{n-2} = q_{n-3}{}^2;$

$$\therefore\ \log\frac{1}{q_{n-3}} = 2^{-2}\,.\,\log\frac{4}{c_n}.$$

Repeat the operation till you get

$$\log\frac{1}{q_1} = 2^{-n+2}\,.\,\log\frac{4}{c_n},$$

next $\quad \log\dfrac{1}{q} = 2^{-n+1}\,.\,\log\dfrac{4}{c_n},$

and $\quad \log\dfrac{1}{\sqrt{q}} = 2^{-n}\log\dfrac{4}{c_n}.$

But this, when $n = \infty$, is the exact value of our ρ. Hence

$$\log \frac{1}{\sqrt{q}} = \rho, \quad = \log \epsilon^{\rho}, \text{ or } q = \epsilon^{-2\rho};$$

an important relation.

29. Assume an arc ω such that

$$\sqrt{b} \tan \omega = \frac{\Lambda}{\Lambda^0}; \quad (1)$$

then ω is always real, and $\omega = 0$ when $x = 0$. While x is $< \frac{1}{2}\pi$, $\tan \omega$ is finite. When x reaches $\frac{1}{2}\pi$, Λ^0 or $\Lambda\,(q, \frac{1}{2}\pi - x)$ vanishes, and Λ or $\Lambda\,(q, x)$ becomes $\Lambda\,(q, \frac{1}{2}\pi)$, so that $\tan \omega$ becomes infinite, and ω reaches $\frac{1}{2}\pi$ simultaneously with x. Thus the two *begin* and *end* every quadrant together.

The combination of

$$\sqrt{b} \tan \omega = \frac{\Lambda}{\Lambda^0},$$

with the two equations

$$\Lambda^2 + b\Lambda^{02} = c\Theta^2; \text{ and } \Theta^2 - b\Theta^2 = c\Lambda^2;$$

makes a system equivalent to the three,

$$\sqrt{c} \sin \omega = \frac{\Lambda}{\Theta}; \quad \sqrt{c} \cos \omega = \sqrt{b} \frac{\Lambda^0}{\Theta}; \quad \Delta\,(c\omega) = \sqrt{b} \,.\, \frac{\Theta^0}{\Theta}; \quad (2).$$

THEOREM. These equations express Elliptic functions of c and ω (modulus and *amplitude*) by functions of q and x.

In Art. 24 where X *meant*

$$\Lambda^0 \,.\, \frac{d\Lambda}{dx} - \Lambda \,.\, \frac{d\Lambda^0}{dx},$$

or
$$\frac{X}{(\Lambda^0)^2} = \frac{d}{dx} \cdot \frac{\Lambda}{\Lambda^0};$$

we had
$$X = (\Lambda \,.\, \tfrac{1}{2}\pi)^2 \,.\, \Theta \,.\, \Theta^0.$$

Thus
$$\frac{X}{(\Lambda^0)^2} = (\Lambda \,.\, \tfrac{1}{2}\pi)^2 \,.\, \frac{\Theta}{\Lambda^0} \cdot \frac{\Theta^0}{\Lambda^0}.$$

Of the left member the equivalent is $\dfrac{d}{dx} \cdot \dfrac{\Lambda}{\Lambda^0}$,

or
$$\frac{d}{dx} \sqrt{b} \tan \omega, \text{ or } \sqrt{b} \sec^2 \omega \,.\, \frac{d\omega}{dx},$$

while on the right, $\dfrac{\Theta}{\Lambda^0} = \dfrac{\sqrt{b}}{\sqrt{c} \cos \omega}$; $\quad \dfrac{\Theta^0}{\Lambda^0} = \dfrac{\Delta\,(c\omega)}{\sqrt{c} \cos \omega}$.

Multiply the equals by $\cos^2\omega$, then

$$\sqrt{b}\,.\,\frac{d\omega}{dx} = (\Lambda\,.\,\tfrac{1}{2}\pi)^2\,.\,\frac{\sqrt{b}\,.\,\Delta(c\omega)}{c},$$

or
$$c\frac{d\omega}{\Delta(c\omega)} = (\Lambda\,.\,\tfrac{1}{2}\pi)^2\,.\,dx.$$

Integrate, then
$$c\,.\,F(c\omega) = (\Lambda\,.\,\tfrac{1}{2}\pi)^2\,.\,x.$$

Thus the Elliptic Integral F is attained.

COR. 1. Let $x = \frac{1}{2}\pi$, then $\omega = \frac{1}{2}\pi$, and

$$c\,.\,(\tfrac{1}{2}\pi\,.\,C) = (\Lambda\,.\,\tfrac{1}{2}\pi)^2\,.\,\tfrac{1}{2}\pi;$$

or
$$cC = (\Lambda\,.\,\tfrac{1}{2}\pi)^2.$$

Eliminate
$$(\Lambda\,.\,\tfrac{1}{2}\pi)^2;$$

then
$$F(c\omega) = Cx \quad (3).$$

This equation *solves for x*, if any one of the four in (1) and (2) be given.

Finally then, we have proved that x is the *Mesonome* to ω.

COR. 2. Since in Art. 26, $\sqrt{c} = \dfrac{\Lambda^0}{\Theta^0}$ with $x = 0$, that is

$$\sqrt{c} = \frac{\Lambda\,.\,\frac{1}{2}\pi}{\Theta\,.\,\frac{1}{2}\pi},$$

whence
$$c\,.\,(\Theta\,.\,\tfrac{1}{2}\pi)^2 = (\Lambda\,.\,\tfrac{1}{2}\pi)^2.$$

We have just shown that $(\Lambda.\,\frac{1}{2}\pi)^2 = cC$,

hence
$$(\Theta\,.\,\tfrac{1}{2}\pi)^2 = C$$

COR. 3. From Art. 26 $\sqrt{b} = \dfrac{\Theta(0)}{\Theta(\frac{1}{2}\pi)}$;

$$\therefore\ \Theta(0) = \sqrt{b}\,.\,\Theta(\tfrac{1}{2}\pi) = \sqrt{b}\,.\,\sqrt{C}.$$

In a single view

$$\Theta\,.\,(\tfrac{1}{2}\pi) = \sqrt{C};\quad \Theta(0) = \sqrt{(bC)};\quad \Lambda(\tfrac{1}{2}\pi) = \sqrt{(cC)}.$$

It is interesting now to observe that

$$\sqrt{C} = \Theta(\tfrac{1}{2}\pi) = 1 + 2q^{1.1} + 2q^{2.2} + 2q^{3.3} + \&c.$$

COR. 4. We now have $\Lambda'(0)$ which by Art. 15 COR.

$$= \Lambda(\tfrac{1}{2}\pi)\,.\,\Theta(0)\,.\,\Theta(\tfrac{1}{2}\pi),$$

is now reduced to
$$\Lambda'(0) = \sqrt{(bcC^3)}.$$

30. Reverting to the four equations which close Art. 23, divide the 3rd by the 1st, then

$$\frac{\Lambda(q^2, x+y)}{\Theta(q^2, x+y)} = \frac{\Lambda x \,.\, \Lambda^0 y + \Lambda y \,.\, \Lambda^0 x}{\Theta x\, \Theta^0 y + \Theta y\, \Theta^0 x} \quad (1).$$

What ω is to c and x, such let θ be to c and y; such too let η be to c_1 (of Landen's scale) and $x+y$. Then

$$\frac{\Lambda(x)}{\Theta(x)} = \sqrt{c}\ \sin\omega\,; \quad \frac{\Lambda^0(x)}{\Theta x} = \sqrt{\frac{c}{b}}\,.\cos\omega\,; \quad \frac{\Theta^0(y)}{\Theta(y)} = \frac{\Delta(c\theta)}{\sqrt{b}}\,,$$

$$\frac{\Lambda^0(y)}{\Theta(y)} = \sqrt{\frac{c}{b}}\,.\cos\theta; \quad \frac{\Lambda(y)}{\Theta(y)} = \sqrt{c}\ \sin\theta\,; \quad \frac{\Theta^0(x)}{\Theta(x)} = \frac{\Delta(c\omega)}{\sqrt{b}}\,.$$

Divide the numerator and denominator of the right member of (1) by $\Theta(x)$, $\Theta(y)$, and equation (1) is transformed to

$$\sqrt{c_1}\,.\,\sin\eta = \frac{\sqrt{c}\sin\omega\sqrt{c}\cos\theta + \sqrt{c}\sin\theta\,.\sqrt{c}\cos\omega}{\Delta(c\theta) + \Delta(c\omega)}$$

$$= \frac{c\sin(\omega+\theta)}{\Delta(c\theta)+\Delta(c\omega)}\,.$$

Such then is the Trigonometrical relation uniting η, ω and θ, when the transcendental relation is

$$\frac{F(c_1\eta)}{c_1} = x + y = \frac{F(c\omega)}{c} + \frac{F(c\theta)}{c}\,.$$

This conclusion embraces Euler's Integrals of Ch. IV., and the scales of Lagrange and Gauss given in Ch. V. We have not *assumed* any of these in the course of our proof. We deduce them as follows.

Cor. 1. Make $\omega = \theta$; therefore

$$\frac{F(c_1\eta)}{c_1} = 2\,.\frac{F(c\omega)}{c}\,, \quad \text{if } \sqrt{c}\sin\eta = \frac{c\sin 2\omega}{2\Delta(c\omega)}\,.$$

This is the scale of Lagrange.

Cor. 2. Make $\theta = 0$, therefore

$$\frac{F(c_1\eta)}{c_1} = \frac{F(c\omega)}{c}\,, \quad \text{if } \sqrt{c_1}\sin\eta = \frac{c\sin\omega}{1+\Delta(c\omega)}\,.$$

This is the scale of Gauss: for it gives

$$\sqrt{c_1}\sin\eta = \sqrt{\frac{1-\Delta(c\omega)}{1+\Delta(c\omega)}}\,.$$

COR. 3. Let $\quad F(c\zeta) = F(c\omega) + F(c\theta).$

Euler's problem was to find the relation of ζ to ω and θ. Here

$$\frac{F(c_1\eta)}{c_1} = \frac{F(c\zeta)}{c}.$$

Then by Cor. 2,

$$\sqrt{c_1}\sin\eta = \sqrt{\frac{1-\Delta(c\zeta)}{1+\Delta(c\zeta)}}.$$

Eliminate η; then

$$\frac{c\sin(\omega+\theta)}{\Delta\theta+\Delta\omega} = \sqrt{\frac{1-\Delta\zeta}{1+\Delta\zeta}};$$

which is identical with Euler's solution.

31. If you change x into $(\frac{1}{2}\pi - x)$, $\frac{\Lambda}{\Theta}$ changes to $\frac{\Lambda^0}{\Theta^0}$ which

$$= \frac{\sqrt{c}\cos\omega}{\Delta(c\omega)};$$

that is, it changes $\sqrt{c}\sin\omega$ to $\frac{\sqrt{c}\cos\omega}{\Delta(c\omega)}$, or exactly $\sin\omega$ to $\sin\omega^0$. Thus to change x to its *Complement* changes ω to its Conjugate. But indeed this was proved in proving that x is the Mesonome to ω.

Relation of $\urcorner$ to the Second Synnomus.

32. Above we found

$$\sqrt{b} = \frac{\Theta(0)}{\Theta(\frac{1}{2}\pi)} = \frac{(\Theta\,.\,0)^2}{\Theta(0)\,.\,\Theta(\frac{1}{2}\pi)} = \left\{\frac{\Theta\,.\,(q,\ 0)^2}{\Theta(q^2,\ 0)}\right\}^2,$$

also $$\Delta(c\omega) = \sqrt{b}\,.\,\frac{\Theta^0 x}{\Theta\,.\,x} = \sqrt{b}\,.\,\frac{\Theta(q^2,\ 0)\,\Theta(q^2,\ 2x)}{[\Theta(q,\ x)]^2}.$$

Give to $\sqrt{b}$ its value from the first line, therefore

$$\Delta(c\omega) = \frac{[\Theta(q,\ 0)]^2}{\Theta(q^2,\ 0)}\,.\,\frac{\Theta(q^2,\ 2x)}{[\Theta(q,\ x)]^2}.$$

For conciseness, let f be short for $\frac{\Theta(q,\ x)}{\Theta(q,\ 0)}$, and when q changes to q^2 let x become $2x$, so that f_1 means

$$\frac{\Theta(q^2,\ 2x)}{\Theta(q^2,\ 0)}.$$

Then $$\Delta(c\omega) = \frac{f_1}{(f)^2},$$

which makes $$-\tfrac{1}{2}\log\Delta(c\omega) = \log f - \tfrac{1}{2}\log f_1.$$

But in Lagrange's scale we found

$$\daleth - \tfrac{1}{2}\daleth_1 = -\tfrac{1}{2}\log\Delta(c\omega).$$

Eliminate $\log\Delta$, then

$$\daleth - \tfrac{1}{2}\daleth_1 = \log f - \tfrac{1}{2}\log f_1,$$

or $$\daleth - \log f = 2^{-1}\{\daleth_1 - \log f_1\}.$$

This equation may be indefinitely repeated by changing q, x into q^2, $2x$; so that we attain

$$\daleth - \log f = 2^{-n}\{\daleth_n - \log f_n\}.$$

If the successive $q\, q^2 q^4 q^8 \ldots$ be denoted by $q\, q_1\, q_2\, q_3 \ldots$ it is clear that q_n is infinitesimal

$$\Theta(q_n x_n) = 1, \quad f_n = 1, \quad \log f_n = 0.$$

Also $\daleth_n = 0$. Hence $$\daleth - \log f = 0,$$

or $$\daleth(c\omega) = \log\frac{\Theta(qx)}{\Theta(q0)}.$$

This proof is Legendre's simplified.

COR. Differentiate: then

$$CG(c\omega) = \frac{d}{dx}\cdot\log\Theta(qx).$$

CHAPTER VIII.

New Elliptic Series.

1. Take the logarithm of the Pronomi

$$\log S = \log (v - v^{-1}) + \Sigma \log (1 - q^{2n} . v^{\pm 2}),$$
$$\log T = \Sigma \log (1 - q^{2n-1} v^{\pm 2}),$$

where $2n$ represents all the even numbers, $2n-1$ all the odd numbers.

Expand the Binomial logarithms by

$$\log (1 - u) = -(u + \tfrac{1}{2}u^2 + \tfrac{1}{3}u^3 + \&c.)$$

and collect the coefficients of

$$v^{\pm 2}, \quad v^{\pm 4}, \quad v^{\pm 6}, \; \&c.$$

by the formulae $k + k^3 + k^5 + k^7 + \&c. = \dfrac{k}{1-k^2}$,

$$k^2 + k^4 + k^6 + k^2 + \&c. = \frac{k^2}{1-k^2}.$$

Then

$$-\log S(v) = -\log (v - v^{-1}) + \frac{q^2(v^2 + v^{-2})}{1-q^2} + \frac{q^4}{2} . \frac{v^4 + v^{-4}}{1-q^4} + \frac{q^6}{3} . \frac{v^6 + v^{-6}}{1-q^6} + \&c.$$

$$-\log T(v) = q . \frac{v^2 + v^{-2}}{1-q^2} + \frac{q^2}{2} . \frac{v^4 + v^{-4}}{1-q^4} + \frac{q^3}{3} . \frac{v^6 + v^{-6}}{1-q^6} + \&c.$$

But again, Art. 8, equation (f) of Ch. VII.

$$\log . q^{\frac{1}{4}} + \log S(v) = \log (Q \sqrt{-1}) + \log . \Lambda (x);$$
$$\log T(v) = \log Q + \log \Theta (x),$$

where

$$v = \epsilon^{x\sqrt{-1}}, \quad v^r + v^{-r} = 2 \cos . rv; \quad v - v^{-1} = 2\sqrt{-1} . \sin v.$$

Hence

$$\left.\begin{aligned}\log Q + \log \Lambda(x) &= \log(2q^{\frac{1}{4}} \sin x) - \frac{2q^2 \cos 2x}{1-q^2} - \frac{2q^4 \cos 4x}{2(1-q^4)} \\ &\qquad - \frac{2q^6 \cos 6x}{3(1-q^6)} - \&c., \\ \log Q + \log \Theta(x) &= -\frac{2q \cos 2x}{1-q^2} - \frac{q^2}{2} \cdot \frac{2\cos 4x}{1-q^4} - \frac{q^3}{3} \cdot \frac{2\cos 6x}{1-q^6} - \&c.\end{aligned}\right\}$$

2. Change x to $\frac{1}{2}\pi - x$, $\Lambda\Theta$ to $\Lambda^0\Theta^0$; then between any *two* of the *four* you can eliminate $\log Q$. Especially we may combine

$$\log \frac{\Lambda}{\Lambda_0} = \log(\sqrt{b} \tan \omega)\,; \quad \log \frac{\Lambda}{\Theta} = \log(\sqrt{c} \sin \omega)\,;$$

$$\log \frac{\Theta^0}{\Theta} = \log \frac{\Delta\omega}{\sqrt{b}}\,; \text{ and } \log \frac{\Lambda^0}{\Theta} = \log\left(\sqrt{\frac{c}{b}} \cos \omega\right).$$

Thus

$$\left[\begin{aligned}&\log(\sqrt{b} \tan \omega) = \log \tan x - \frac{4q^2 \cos 2x}{1-q^2} - \frac{4q^6 \cos 6x}{3(1-q^6)} - \frac{4q^{10} \cos 10x}{5(1-q^{10})} - \&c., \\ &\log(\sqrt{c} \sin \omega) = \log(2q^{\frac{1}{4}} \sin x) + \frac{2q \cos 2x}{1+q} + \frac{2q^2 \cos 4x}{2(1+q^2)} \\ &\qquad\qquad + \frac{2q^3 \cos 6x}{3(1+q^3)} + \&c., \\ &\log \frac{\Delta(\omega)}{\sqrt{b}} = \frac{4q \cos 2x}{1-q^2} + \frac{4q^3 \cos 6x}{3(1-q^6)} + \frac{4q^5 \cos 10x}{5(1-q^{10})} + \&c., \\ &\log\left(\sqrt{\frac{c}{b}} \cos \omega\right) = \log(2q^{\frac{1}{4}} \cos x) \\ &\qquad + \frac{2(q+q^2) \cos 2x}{1-q^2} + \frac{2(q^2-q^4) \cos 4x}{1-q^4} + \frac{2(q^3+q^6) \cos 6x}{1-q^6} - \&c.,\end{aligned}\right.$$

where in the last

$$\frac{q+q^2}{1-q^2} = \frac{q}{1-q}\,; \quad \frac{q-q^4}{1-q^4} = \frac{q^2}{1+q^2}\,; \quad \frac{q^3+q^6}{1-q^6} = \frac{q^3}{1-q^3}\,; \ \ldots$$

so that with alternate $1 \pm q^n$ in the denominators

$$\log\left(\sqrt{\frac{c}{b}} \cos \omega\right) = \log(2q^{\frac{1}{4}} \cos x)$$

$$+ \frac{2q \cos 2x}{1-q} + \frac{2q^2 \cos 4x}{2(1+q^2)} + \frac{2q^3 \cos 6x}{3(1-q^3)} + \frac{2q^4 \cos 8x}{4(1+q^4)} + \&c.$$

In the second and fourth of these series, $q^{\frac{1}{4}}$ meets us in the first term, under *log*. Now we have found Ch. VII. Art. 28,

$$q = \epsilon^{-2\rho}, \quad \therefore\ q^{\frac{1}{4}} = \epsilon^{-\frac{1}{2}\rho} \text{ and } \log q^{\frac{1}{4}} = -\tfrac{1}{2}\rho;$$

$$\therefore\ \log(\surd c \sin\omega) = +\log 2 - \tfrac{1}{2}\rho + \log\sin x$$
$$+\frac{2q\cos 2x}{1+q} + \frac{2q^2\cos 4x}{2(1+q^2)} + \frac{2q^3\cos 6x}{3(1+q^3)} + \&c.,$$

$$\log\left(\sqrt{\frac{c}{b}}\cos\omega\right) = \log 2 - \tfrac{1}{2}\rho + \log\cos x$$
$$+\frac{2q\cos 2x}{1-q} + \frac{2q^2\cos 4x}{2(1+q^2)} + \frac{2q^3\cos 6x}{3(1-q^3)} + \frac{2q^4\cos 8x}{4(1-q^4)} + \&c.$$

3. In the four series of the last Article, make $\omega = 0 = x$, then we have

$$\omega = Cx, \quad \frac{\tan\omega}{\tan x} = \frac{\sin\omega}{\sin x} = \frac{\omega}{x} = C.$$

Hence

$$\left[\begin{aligned}
&\log(\surd bC) = -\frac{4q^2}{1-q^2} - \frac{4q^6}{3(1-q^6)} - \frac{4q^{10}}{3(1-q^{10})} - \&c.,\\
&\log(\surd cC) = \log 2 - \tfrac{1}{2}\rho + \frac{2q}{1+q} + \frac{2q^2}{2(1+q^2)} + \frac{2q^3}{3(1+q^6)} + \&c.,\\
&\log\frac{1}{\surd b} = \frac{4q}{1-q^2} + \frac{4q^3}{3(1-q^6)} + \frac{4q^5}{5(1-q^{10})} + \&c.,\\
&\log\sqrt{\frac{c}{b}} = \log 2 - \tfrac{1}{2}\rho + \frac{2q}{1-q} + \frac{2q^2}{2(1+q^2)} + \frac{2q^3}{3(1-q^3)}\\
&\qquad\qquad + \frac{2q^4}{4(1+q^4)} + \&c.
\end{aligned}\right.$$

Subtract each result from its own series, and our four equations appear as follows:

$$\left[\begin{aligned}
&\log\frac{\tan\omega}{C} = \log\tan x + \frac{4q^2(1-\cos 2x)}{1-q^2} + \frac{4q^6(1-\cos 6x)}{3(1-q^6)} + \&c.,\\
&\log\frac{\sin\omega}{C} = \log\sin x - \frac{2q(1-\cos 2x)}{1+q} - \frac{2q^2(1-\cos 4x)}{2(1+q^2)}\\
&\qquad\qquad - \frac{2q^3(1-\cos 6x)}{3(1+q^3)} - \&c.\\
&-\log\Delta(c\omega) = \frac{4q(1-\cos 2x)}{1-q^2} + \frac{4q^3(1-\cos 6x)}{3(1-q^6)} + \frac{4q^5(1-\cos 10x)}{5(1-q^{10})} + \&c.,\\
&\log.\cos\omega = \log\cos x - \frac{2q(1-\cos 2x)}{1-q} - \frac{2q^2(1-\cos 4x)}{2(1+q^2)} - \&c.
\end{aligned}\right.$$

These four may now be differentiated, observing that

$$d\omega = C \,.\, \Delta\,(c\,\omega) \,.\, dx,$$

by aid of which we eliminate $d\omega$ and dx.

4. Thus we obtain:

$$\left[\begin{aligned}
\frac{C\,.\,\Delta}{\sin\omega\cos\omega} &= \frac{1}{\sin x\cos x} + \frac{8q^2\sin 2x}{1-q^2} + \frac{8q^6\sin 6x}{1-q^6} + \&c.\\
C\,.\cot\omega\,.\,\Delta &= \cot x - \frac{4q\,.\sin 2x}{1+q} - \frac{4q^2\sin 4x}{1+q^2} - \&c.\\
\frac{Cc^2\,.\sin\omega\cos\omega}{\Delta} &= \frac{8q\sin 2x}{1-q^2} + \frac{8q^3\sin 6x}{1-q^6} + \frac{8q^5\sin 10x}{1-q^{10}} + \&c.\\
C\tan\omega\,.\,\Delta &= \tan x + \frac{4q\sin 2x}{1-q} + \frac{4q^2\sin 4x}{1+q^2} + \frac{4q^3\sin 6x}{1-q^3} + \&c.
\end{aligned}\right.$$

But by help of Lagrange's scale and Gauss's obverse of it, the three first of these are simplified; and the fourth by Euler's Duplication.

Namely in Lagrange's scale the relation

$$\sqrt{c_1}\,\sin\omega_1 = \frac{c\sin\omega\cos\omega}{\Delta\,(c\omega)}$$

gives both

$$\frac{C\Delta}{\sin\omega\cos\omega} = \frac{2C_1}{\sin\omega_1} \text{ and } Cc^2\,.\frac{\sin\omega\cos\omega}{\Delta} = 2C_1c_1\sin\omega_1,$$

which changes our first and third just obtained into

$$\left[\begin{aligned}
\frac{C_1}{\sin\omega_1} &= \frac{1}{\sin 2x} + \frac{4q^2\sin 2x}{1-q^2} + \frac{4q^6\sin 6x}{1-q^6} + \frac{4q^{10}\sin 10x}{1-q^{10}} + \&c.\\
C_1c_1\sin\omega_1 &= \frac{4q\sin 2x}{1-q^2} + \frac{4q^3\sin 6x}{1-q^6} + \frac{4q^5\sin 10x}{1-q^{10}} + \&c.
\end{aligned}\right.$$

in which to change $C_1\,c_1\,\omega_1$ to $C\,c\,\omega$, is justified by changing q to $\sqrt{q}$, x to $\frac{1}{2}x$.

Deal with the second of our four equations by the scale of Gauss,

$$\frac{F\,(c_1\eta)}{C_1} = x = \frac{F\,(c\omega)}{C}$$

which makes $C_1\cot\eta\,.\,\Delta\,(c_1\eta) = C\cot\omega$. Replace $c\omega$ by $c_1\eta$ in the second series, then, *without change of x*, we must replace q by q^2, making

$$C_1\cot\omega_1\,.\,\Delta\,(c_1\eta) = \cot x - \frac{4q^2\sin 2x}{1+q^2} - \frac{4q^4\sin 4x}{1+q^4} - \frac{4q^6\sin 6x}{1+q^6} - \&c.$$

in which the left member $= C\cot\omega$, the right hand remaining unchanged.

We modify the fourth equation by assuming

$$F(c\eta) = 2F(c\omega), \ \therefore \ \tan(\tfrac{1}{2}\eta) = \tan\omega\,.\,\Delta\omega,$$

then since $\left.\begin{array}{l} F(c\omega) = Cx_1 \\ F(c\eta) = Cx' \end{array}\right\}$ we have $x' = 2x, \ x = \tfrac{1}{2}x'$;

$$\therefore \ C\tan(\tfrac{1}{2}\eta) = \tan(\tfrac{1}{2}x') + \frac{4q\sin x'}{1-q} + \frac{4q^2\sin 2x'}{1+q^2} + \frac{4q^3\sin 3x'}{1-q^3} + \&c.,$$

in which we may now replace $\eta\, x'$ by $\omega\, x$ for uniformity.

Finally in place of our four equations we have as equivalents, by changing in first and third $c_1C_1\omega_1$ into $cC\omega$, therefore q^2, $2x$ into q, x; in second and fourth as just stated: therefore

$$\left[\begin{array}{l} \dfrac{C}{\sin\omega} = \dfrac{1}{\sin x} + \dfrac{4q\sin x}{1-q} + \dfrac{4q^3\sin 3x}{1-q^3} + \dfrac{4q^5\sin 5x}{1-q^5} + \&c., \\ C\cot\omega = \cot x - \dfrac{4q^2\sin 2x}{1+q^2} - \dfrac{4q^4\sin 4x}{1+q^4} - \dfrac{4q^6\sin 6x}{1+q^6} - \&c., \\ Cc\sin\omega = \dfrac{4\sqrt{q}\sin x}{1-q} + \dfrac{4\sqrt{q^3}\sin 3x}{1-q^3} + \dfrac{4\sqrt{q^5}\sin 5x}{1-q^5} + \&c., \\ C\tan(\tfrac{1}{2}\omega) = \tan(\tfrac{1}{2}x) + \dfrac{4q\sin x}{1-q} + \dfrac{4q^2\sin 2x}{1+q^2} + \dfrac{4q^3\sin 3x}{1-q^3} + \&c. \end{array}\right.$$

5. In the first and third of this last set, change q into $-q$, leaving x unchanged and pursue the further consequences. For $\sqrt{q}, \sqrt{q^3}, \sqrt{q^5}\ldots$ we must replace the same quantities multiplied by $\sqrt{-1}, -\sqrt{-1}, +\sqrt{-1},\ldots$ For Λ we must write $\sqrt[4]{-1}\,.\,\Lambda$; Θ is changed to Θ^0, b or $\left\{\dfrac{\Theta\,.\,(q,0)}{\Theta\,.\,(q,\frac{1}{2}\pi)}\right\}^2$ becomes inverted, or changes to b^{-1}; hereby changing c to $cb^{-1}\sqrt{-1}$. Also because $\sqrt{C} = \Theta(q, \tfrac{1}{2}\pi)$, $\sqrt{C}$ now becomes $\Theta(-q, \tfrac{1}{2}\pi)$ or $\Theta(q, 0)$ or $\sqrt{(bC)}$, $\therefore$ C changes to bC.

Hence $Cc\sin\omega$ or $(C\sqrt{c})(\sqrt{c}\sin\omega)$ becomes

$$(bC)\left(\sqrt{\frac{c}{b}}\sqrt[4]{-1}\right)\,.\,\frac{\Lambda\sqrt[4]{-1}}{\Theta^0}.$$

But
$$\frac{\Lambda}{\Theta^0} = \sqrt{(bc)}\,.\,\frac{\sin\omega}{\Delta(c\omega)};$$

or $Cc\sin\omega$ becomes $bC\sqrt{\dfrac{c}{b}}\,.\,\sqrt{-1}\,.\,\sqrt{(bc)}\,\dfrac{\sin\omega}{\Delta(c\omega)}$.

Again $\dfrac{C}{\sin\omega}$ or $C\sqrt{c}$ divided by $\sqrt{c}\sin\omega$ changes to $bC\,.\sqrt{\dfrac{c}{b}}\,\sqrt[4]{-1}$ divided by

$$\sqrt[4]{-1}\,.\,\sqrt{(bc)}\,.\,\frac{\sin\omega}{\Delta\,(c\omega)} \text{ or } \frac{C\Delta\,(c\omega)}{\sin\omega}\,.$$

Thus our first and third series become (when the third is divided by $\sqrt{-1}$)

$$\left.\begin{aligned}\frac{C\Delta(c\omega)}{\sin\omega} &= \frac{1}{\sin x} - \frac{4q\sin x}{1+q} - \frac{4q^3\sin 3x}{1+q^3} - \frac{4q^5\sin 5x}{1+q^5} - \&\text{c.},\\ Cc\,.\,\frac{b\sin\omega}{\Delta\,(c\omega)} &= \frac{4\sqrt{q}\sin x}{1+q} - \frac{4\sqrt{q^3}\sin 3x}{1+q^3} + \frac{4\sqrt{q^5}\sin 5x}{1+q^5} - \&\text{c.}\end{aligned}\right\}$$

In each of these change x to $(\frac{1}{2}\pi - x)$ then ω changes to its Conjugate ω^0, and $\dfrac{b\sin\omega}{\Delta\,(c\omega)}$ being $=\cos\omega^0$, changes to $\cos\omega$.

Finally

$$\left.\begin{aligned}\frac{bC}{\cos\omega} &= \frac{1}{\cos x} - \frac{4q\cos x}{1+q} + \frac{4q^3\cos 3x}{1+q^3} - \frac{4q^5\cos 5x}{1+q^5} + \&\text{c.},\\ Cc\cos\omega &= \frac{4\sqrt{q}\cos x}{1+q} + \frac{4\sqrt{q^3}\cos 3x}{1+q^3} + \frac{4\sqrt{q^5}\cos 5x}{1+q^5} + \&\text{c.},\end{aligned}\right\}$$

an important addition to the previous four equations.

Further, for the penultimate we may write

$$\frac{bC}{\cos\omega^0} = \frac{1}{\sin x} - \frac{4q\sin x}{1+q^2} - \frac{4q^3\sin 3x}{1+q^3} - \frac{4q^5\sin 5x}{1+q^5} - \&\text{c.}$$

Observe that
$$\frac{b}{\cos\omega^0} = \frac{\Delta\,(c\omega)}{\sin\omega}\,;$$

also
$$\frac{d\omega}{C\Delta\,(c\omega)} = dx,$$

where

$$\frac{d\omega}{\sin\omega} = \frac{dx}{\sin x} - \frac{4q\sin x\,dx}{1+q^2} - \frac{4q^3\sin 3x\,dx}{1+q^3} - \frac{4q^5\sin 5x\,dx}{1+q^5} - \&\text{c.}$$

Integrate: then

$$\log\tan.\tfrac{1}{2}\omega = \log\tan.\tfrac{1}{2}x + \frac{4q\cos x}{1+q^2} + \frac{4q^3.\cos 3x}{3\,(1+q^3)} + \frac{4q^5.\cos 5x}{5\,(1+q^5)} + \&\text{c.},$$

when
$$\omega = 0 = x, \quad \omega = Cx.$$

Also $\quad \log.\dfrac{\tan\frac{1}{2}\omega}{\tan\frac{1}{2}x}$ converges to $\log\dfrac{\omega}{x}$ or $\log C$,

so that
$$\log C = \frac{4q}{1+q^2} + \tfrac{1}{3}\,.\,\frac{4q^3}{1+q^3} + \tfrac{1}{5}\,.\,\frac{4q^5}{1+q^5} + \&\text{c.}$$

6. We can now, by aid of the last, deduce a series for $\Delta(c\omega)$ in terms of x and q by Lagrange's scale, which has (Ch. v. Art. 2)

$$C\Delta - C_1\Delta_1 = c_1C_1 \cos\omega_1.$$

The last by repetition yields

$$C\Delta - C_n\Delta_n = C_1c_1 \cos\omega_1 + C_2c_2 \cos\omega_2 + \ldots + C_nc_n \cos\omega_n.$$

Let x be infinite, then $C_n\Delta_n$ converges to $\sqrt{(1 - c_n^2 \sin^2\omega_n)}$ or 1, and

$$C\Delta(c\omega) - 1 = C_1c_1 \cos\omega_1 + C_2c_2 \cos\omega_2 + \text{\&c., } ad\ infin.$$

But from Art. 5, we have

$$C_1c_1 \cos\omega_1 = \frac{4q \cos 2x}{1+q^2} + \frac{4q^3 \cos 6x}{1+q^6} + \frac{4q^5 \cos 10x}{1+q^{10}} + \text{\&c.},$$

in which the indices of the terms are 2, 6, 10, 14... or double of the odd numbers. To change $c_1\omega_1$ to $c_2\omega_2$ gives a like form, with indices 2^2 of the odd numbers.

Further change to $c_3\omega_3$... and the indices become 2^3 of the odd numbers; and so on. When all these are added together, the sum

$$\{C\Delta(c\omega) - 1\}$$

has in the collective terms the indices that are double of 1, 2, 3, 4... in complete series, that is

$$C\Delta(c\omega) = 1 + \frac{4q \cos 2x}{1+q^2} + \frac{4q^2 \cos 4x}{1+q^4} + \frac{4q^3 \cos 6x}{1+q^6} + \frac{4q^4 \cos 8x}{1+q^8} + \text{\&c.}$$

Differentiate this; first observing that

$$d\sqrt{(1 - c^2\sin^2\omega)} = \frac{-\frac{1}{2}c^2 \,.\, d \,.\, \sin^2\omega}{\sqrt{(1-c^2\sin^2\omega)}} = \frac{-c^2 \sin\omega \cos\omega \, d\omega}{\Delta(c\omega)}$$

$$= -c^2 \sin\omega \cos\omega \, C\, dx,$$

$$\therefore \frac{d\Delta(c\omega)}{dx} = -Cc^2 \sin\omega\cos\omega = -\tfrac{1}{2}Cc^2 \sin 2\omega.$$

Hence

$$\tfrac{1}{2}C^2c^2 \sin 2\omega = 8\left\{\frac{q \sin 2x}{1+q^2} + \frac{2q^2 \sin 4x}{1+q^4} + \frac{3q^3 \cos 6x}{1+q^6} + \text{\&c}\right\}.$$

7. Again, multiply the series for $C\Delta(c\omega)$ by $\dfrac{d\omega}{C\Delta(c\omega)} = dx$, and integrate; then

$$\omega = x + \frac{2q \sin 2x}{1+q^2} + \frac{2q^2 \sin 4x}{2(1+q^4)} + \frac{2q^3 \sin 6x}{3(1+q^6)} + \frac{2q^4 \sin 8x}{4(1+q^8)} + \text{\&c.}$$

Further, in Lagrange's scale,

$$CG\,(c\omega) = C_1c_1 \sin \omega_1 + C_2c_2 \sin \omega_2\ C_3c_3 \sin \omega_3 + \&c.$$

Replace each term of this series by its equivalent in series of q and x attained above. In the series for $Cc \sin \omega$ we have a type of all, with indices 1, 3, 5, 7.... In forming the equivalent of CG we begin from indices *double* of these odd numbers, and we add terms with indices which multiply these by 2^2, 2^3, 2^4, 2^8... of which the sum is *twice* the complete series 1, 2, 3, 4.... That is

$$CG\,(c\omega) = \frac{4q \sin 2x}{1-q^2} + \frac{4q^2 \sin 4x}{1-q^4} + \frac{4q^3 \sin 6x}{1-q^6} + \frac{4q^4 \sin 8x}{1-q^8} + \&c.$$

This last series may be either *integrated* after multiplying by

$$\frac{dF\,(c\omega)}{c} = dx,$$

whence

$$\daleth\,(c\omega) = \frac{2q\,(1-\cos 2x)}{1-q^2} + \frac{2q^2(1-\cos 4x)}{2\,(1-q^4)} + \frac{2q^3(1-\cos 6x)}{3\,(1-q^6)} + \&c.,$$

else differentiating $CG\,(c\omega)$, observing that

$$dG = dE - \aleph_c\,.\,dF = (\Delta^2 - \aleph_c)\,C dx,$$

$$\therefore\ \tfrac{1}{4}C^2(\Delta^2 - \aleph_c) = \frac{2q \cos 2x}{1-q^2} + \frac{4q^2 \cos 4x}{1-q^4} + \frac{6q^2 \cos 6x}{1-q^6} + \&c.$$

Make $$\omega = 0 = x,$$

$$\therefore\ \tfrac{1}{4}C^2(1 - \aleph_c) = \frac{2q}{1-q^2} + \frac{4q^2}{1-q^4} + \frac{6q^3}{1-q^6} + \&c.$$

Eliminate $\aleph_c$; $$\tfrac{1}{4}C^2(c^2 \sin^2 \omega),$$

or $\frac{1}{8}C^2c^2(1-\cos 2\omega)$

$$= \frac{2q(1-\cos 2x)}{1-q^2} + \frac{4q^2(1-\cos 4x)}{1-q^4} + \frac{6q^3(1-\cos 6x)}{1-q^6} + \&c.$$

8. It is of little importance that from the series for $C \cot \omega$ we may change ω into ω^0 and x into $(\frac{1}{2}\pi - x)$. Then since

$$\cot \omega^0 = b \tan \omega,$$

you find

$$Cb \tan \omega = \tan x - \frac{4q^2 \sin 2x}{1+q^2} + \frac{4q^4 \sin 4x}{1+q^4} - \frac{4q^6 \sin 6x}{1+q^6} + \&c.$$

We verify, in the two last, by Lagrange's scale which gives

$$C_1 \cot \omega_1 = \tfrac{1}{2}C(\cot \omega - b \tan \omega).$$

9. In all these series we may now with advantage pass from functions of q to functions of ρ, which becomes our leading constant. We have

$$\frac{2q}{1+q}=\text{ח}(\rho);\quad \frac{2q}{1-q}=\text{כ}(\rho);\quad \frac{2\sqrt{q}}{1-q}=\frac{1}{\operatorname{Sin}\rho};\quad \frac{2\sqrt{q}}{1+q}=\frac{1}{\operatorname{Cos}\rho};$$

and when q is changed to q^n, ρ changes to $(n\rho)$. This makes the transformation very easy.

With this small change it is well now to recapitulate, from Articles 2, 4 above.

10. First from Art. 2, next from *close* of Art. 4, also from Art. 5, &c.

1. $\log(\sqrt{c}\sin\omega)=\log 2-\frac{1}{2}\rho+\log\sin x+\text{ח}(\rho)\,.\cos 2x$
$+\frac{1}{2}\text{ח}(2\rho)\cos 4x+\frac{1}{3}\text{ח}(3\rho)\cos 6x+\&c.$

2. $\frac{1}{2}C\operatorname{cosec}\omega=\frac{1}{2}\operatorname{cosec}x+\text{כ}(\rho)\sin x+\text{כ}(3\rho)\sin 3x+\text{כ}(5\rho)$
$\sin 5x+\&c.$

3. $\frac{1}{2}C\cot\omega=\frac{1}{2}\cot x-\text{ח}(2\rho)\sin 2x-\text{ח}(4r)\sin 4x-\text{ח}(6\rho)$
$\sin 6x-\&c.$

4. $\frac{1}{2}bC\tan\omega=\frac{1}{2}\tan x-\text{ח}(2\rho)\sin 2x+\text{ח}(4\rho)\sin 4x-\text{ח}(6\rho)$
$\sin 6x+\&c.$

5. $\frac{1}{2}Cc\sin\omega=\dfrac{\sin x}{\operatorname{Sin}\rho}+\dfrac{\sin 3x}{\operatorname{Sin}3\rho}+\dfrac{\sin 5x}{\operatorname{Sin}5\rho}+\&c.$

6. $\frac{1}{2}C\tan\frac{1}{2}\omega=\frac{1}{2}\tan\,.\,\frac{1}{2}x+\text{כ}(\rho)\sin x+\text{ח}(2\rho)\sin 2x+\text{כ}(3\rho)\sin 3x$
$+\text{ח}(4\rho)\sin 4x+\&c.$

7. $\frac{1}{2}Cc\,.\cos\omega=\dfrac{\cos x}{\operatorname{Cos}\rho}+\dfrac{\cos 3x}{\operatorname{Cos}3\rho}+\dfrac{\cos 5x}{\operatorname{Cos}5\rho}+\&c.$

8. $\frac{1}{2}bC\sec\omega=\frac{1}{2}\sec x-\text{ח}(\rho)\cos x+\text{ח}(3\rho)\cos 3x-\text{ח}(5x)$
$\cos 5x+\&c.$

9. $C.\Delta(c\omega)=1+2\,.\,\dfrac{\cos 2x}{\operatorname{Cos}2\rho}+2\,.\,\dfrac{\cos 4x}{\operatorname{Cos}4\rho}+2\,.\,\dfrac{\cos 6x}{\operatorname{Cos}6\rho}+\dots\,.$

10. $\frac{1}{8}C^2c^2\,.\sin 2\omega=\dfrac{\sin 2x}{\operatorname{Cos}2\rho}+\dfrac{2\sin 4x}{\operatorname{Cos}4\rho}+\dfrac{3\sin 6x}{\operatorname{Cos}6\rho}+\&c.$

11. $\omega=x+\dfrac{\sin 2x}{\operatorname{Cos}2\rho}+\dfrac{1}{2}\,.\,\dfrac{\sin 4x}{\operatorname{Cos}4\rho}+\dfrac{1}{3}\,.\,\dfrac{\sin 6x}{\operatorname{Cos}6\rho}+\&c.$

12. $\frac{1}{2}C\,.\,G(c\omega)=\dfrac{\sin 2x}{\operatorname{Sin}2\rho}+\dfrac{\sin 4x}{\operatorname{Sin}4\rho}+\dfrac{\sin 6x}{\operatorname{Sin}6\rho}+\&c.$

13. $$\daleth(c\omega) = \frac{1-\cos 2x}{\text{Sin } 2\rho} = \frac{1}{2} \cdot \frac{1-\cos 4x}{\text{Sin } 4\rho} + \frac{1}{3} \cdot \frac{1-\cos 6x}{\text{Sin } 6\rho} + \&c.$$

14. $$\tfrac{1}{8}C^2 . (1-\cos 2\omega) = \frac{1-\cos 2x}{\text{Sin } 2\rho} + \frac{2\,(1-\cos 4x)}{\text{Sin } 4\rho} + \frac{3\,(1-\cos 6x)}{\text{Sin } 6\rho} + \&c.$$

Chiefly by assuming

$$\omega = \tfrac{1}{2}\pi = x, \quad \text{or} \quad \omega = 0 = x, \quad \text{with} \quad \omega = Cx,$$

we deduce *relations between our constants*, without which we never can command numerical results. When $\tan\omega$ and $\tan x$ enter, the assumption $\omega = \omega^0$ makes $x = \tfrac{1}{2}\pi$ or

$$\tan^2\omega = b^{-1}, \quad \therefore \; -\cos 2\omega = \frac{1-b}{1+b}.$$

We also have (in Art. 29, Cor. 2)

$$\sqrt{C} = \Theta\,(\tfrac{1}{2}\pi).$$

From the four equations in Art. 3 with a logarithm within on the left.

15. $$-\tfrac{1}{2}\log \Delta\,(c\omega) = \frac{1-\cos 2x}{\text{Sin } 2\rho} + \frac{1-\cos 6x}{3 \text{ Sin } 6\rho} + \frac{1-\cos . 10x}{5 \text{ Sin } 10\rho} + \&c.$$

16. $$\tfrac{1}{2}\log\frac{\tan\omega}{C} = \tfrac{1}{2}\log\tan x + \beth\,(2\rho)\,(1-\cos 2x) + \tfrac{1}{3} . \beth\,(6\rho)\,(1-\cos 6x) + \tfrac{1}{5}\beth\,(10\rho)\,(1-\cos 10x) + \&c.$$

17. $$-\log\frac{\sin\omega}{C} = -\log\sin x + ה\,(\rho)\,(1-\cos 2x) + \tfrac{1}{2}ה\,(2\rho)\,(1-\cos 4x) + \tfrac{1}{3}ה\,(3\rho)\,(1-\cos 6x) + \&c.$$

18. $$-\log\cos\omega = -\log\cos x + \beth\,(\rho)\,(1-\cos 2x) + \tfrac{1}{2}\beth\,(2\rho)\,(1-\cos 4x) + \tfrac{1}{3}\beth\,(3\rho)\,(1-\cos 6x) + \&c.$$

Also differentiating No. 5 with

$$d\omega = C . \Delta\,(c,\,\omega)\,dx.$$

19. $$\tfrac{1}{2}Cc\cos\omega . C\Delta\,(c\omega) = \frac{\cos x}{\text{Sin } \rho} + \frac{3\cos 3x}{\text{Sin } 3\rho} + \frac{5\cos 5x}{\text{Sin } 5\rho} + \&c.$$

And from close of Art. 5

20. $$\log\tan\tfrac{1}{2}\omega = \log\tan . \tfrac{1}{2}x + 2\,\{ה\,(\rho)\cos x + \tfrac{1}{3}ה\,(3\rho)\cos 3x + \tfrac{1}{5}ה\,(5\rho)\cos 5x + \&c.\}.$$

This article displays to the eye the eminence of the Anticyclic constants. For shortness I write ק for Cosec but Sec may well stand.

We have already found

$$\log\frac{1}{q}=2\rho\,;\quad \sqrt{c}=\frac{\Lambda\,(\frac12\pi)}{\Theta\,(\frac12\pi)};\quad \sqrt{b}=\frac{\Theta\,(0)}{\Theta\,(\frac12\pi)}\,;$$

which are known functions of ρ. So is

$$\sqrt{C}=\Theta\,(\tfrac12\pi).$$

We also have

$$\sqrt[\frac14]{\frac{1}{b}}=\text{Cot}\,\rho\ .\ \text{Cot}\,3\rho\ .\ \text{Cot}\,5\rho\ \text{\&c.}$$

The method of changing q to $-q$, by which Nos. 7 and 8 were deduced from No. 5 and No. 2, may be tried on others of these 20. Applied to No. 1, it only reproduces

$$\log\left(\sqrt{\frac{c}{b}}\cos\omega^0\right).$$

Applied to No. 9, it brings the same result as by changing

$$\omega \text{ to } \omega^0,\quad x \text{ to } \tfrac12\pi-x.$$

It is difficult to limit these series, yet probably we have attained the most valuable.

11. We seek for more convenient relations by making $x=0$, or $x=\frac12\pi$ in Art. 10.

From No. 1. $\frac{1}{2}\left(\log\frac{4}{c}-\rho\right)=ח\,(\rho)-\frac12 ח(2\rho)+\frac13 ח\,(3\rho)-$ &c.

" " 2. $\frac12 C=\frac12+ב\,(\rho)+ב\,(5\rho)-ב\,(7\rho)+$ &c.

From 3, with $x=\frac14\pi$, $\cot\omega=\sqrt{b}$ } $\frac12 C\sqrt{b}=\frac12-ח\,(2\rho)+ח\,(6\rho)+ח\,(10\rho)+$ &c.

From No. 5. $\frac12 Cc=ק\,(\rho)-ק\,(3\rho)+ק\,(5\rho)-$ &c.

" " 6. $\frac12 C=\frac12+ב\,(\rho)-ב\,(3\rho)+ב\,5\,(\rho)-$ &c. as before.

" " 7. $\frac12 Cc=\text{Sec}\,\rho+\text{Sec}\,3\rho+\text{Sec}\,5\rho+\dots$.

" " 8. $\frac12 bC=\frac12-ח\,(\rho)+ח\,(3\rho)-ח\,(5\rho)+$ &c.

" " 9. Make $x=0$, $\therefore\ C=1+2\,\text{Sec}\,2\rho+2\,\text{Sec}\,4\rho+2\,\text{sec}\,6\rho$.

" " 10. $^*\frac{1}{8}\,.\,C^2c^2\,.\,\frac{2\sqrt{b}}{1+b}=\text{Sec}\,(2\rho)-3\,\text{Sec}\,(6\rho)+5\,\text{Sec}\,(10\rho)\,;$

* In the result from 10, writing b_1 for $\frac{2\sqrt{b}}{1+b}$, and $C_1{}^2c_1$ for $\frac14 C_2c^2$, we may pass from $c_1\ 2\rho\ b_1$ to $c\ \rho\ b$, then $\frac12 C^2cb=\text{Sec}\,\rho-3\,\text{Sec}\,(3\rho)+5\,\text{Sec}\,(5\rho)-$ &c.

but $$\frac{2\sqrt{b}}{1+b}=b_1.$$

From No. 14. $\frac{1}{8}C^2=\text{ק}(2\rho)+3\text{ק}(6\rho)+5\text{ק}(10\rho)+\&c.$

These results from 9 add 7 are verified by

$$C_1+C_1c_1=C.$$

In 9 make $x=\frac{1}{2}\pi$, therefore

$$Cb=1-2\,\mathrm{Sec}\,2\rho+2\,\mathrm{Sec}\,4\rho-2\,\mathrm{Sec}\,6\rho+\&c.$$

but make

$$x=\tfrac{1}{4}\pi,\ \cot^2\omega=b,\ \sin^2\omega=(1+b)^{-1},\ c^2\sin^2\omega=1-b,\ \Delta(c\omega)=\sqrt{b}\,;$$

$$\therefore\ C\sqrt{b}=1-2\,\mathrm{Sec}\,4\rho+2\,\mathrm{Sec}\,8\rho-2\,\mathrm{Sec}\,12\rho+\&c.$$

(Evidently, since $bC=\sqrt{b_1}C_1$.)

So in 15 making $\quad n=\frac{1}{4}\pi,\quad \Delta(c\omega)=\sqrt{b}.$

$$\frac{1}{4}\log\frac{1}{b}=\text{ק}(2\rho)+\tfrac{1}{3}\text{ק}(6\rho)+\tfrac{1}{5}\text{ק}(10\rho)+\&c.$$

From Art. 7 above, we have also

$$\tfrac{1}{4}C^2.(1-\aleph_c)=\text{ק}(2\rho)+2\text{ק}(4\rho)+3\text{ק}(6\rho)+\&c.$$

Moreover from Chapter II. Art. 7, we have

$$\log\frac{4}{c}-\rho=\log C-\tfrac{1}{2}\log C_1-\tfrac{1}{4}\log C_2-\tfrac{1}{8}\log C_3-\&c\,,$$

which usefully checks the first equation of this Article, and with higher convergence, when we know $\log C$.

Make $\omega=0=x$ in No. 19, therefore

$$\tfrac{1}{2}C^2c=\text{ק}(\rho)+3\text{ק}(3\rho)+5\,(5\rho)+\&c.$$

12. In No. 17 of Art. 10 make $\omega=\frac{1}{2}\pi=x$, therefore

$$\tfrac{1}{2}\log C=\text{ח}(\rho)+\tfrac{1}{3}\text{ח}(3\rho)+\tfrac{1}{5}\text{ח}(5\rho)+\&c.$$

Also from No. 20 we have the very same.

We also had $\quad \sqrt{C}=\dfrac{\mathrm{Cot}\,\rho}{\mathrm{Cot}\,2\rho}\cdot\dfrac{\mathrm{Cot}\,3\rho}{\mathrm{Cot}\,4\rho}\cdot\dfrac{\mathrm{Cot}\,5\rho}{\mathrm{Cot}\,6\rho}\cdot\&c.,$

whence $\quad \frac{1}{2}\log C=l\,\mathrm{Cot}\,\rho-l\,\mathrm{Cot}\,2\rho+l\,\mathrm{Cot}\,3\rho-l\,\mathrm{Cot}\,4\rho+\&c.$

[Again in Ch. II. Art. 4, we had

$$\sqrt{(bC)} = \sqrt[4]{(bb_1\, b_2\, b_3 \ldots)}.$$

But $$\sqrt[4]{b} = (\text{Cot}\,\rho\,.\,\text{Cot}\,3\rho\,.\,\text{Cot}\,5\rho\,\ldots)^{-1},$$

by which we replace the factors of the left in the penultimate by observing that to add a unit to the index of b doubles the pro-modulus ρ.

Thus $$\sqrt[4]{b_1} = (\text{Cot}\,2\rho\,.\,\text{Cot}\,6\rho\,.\,\text{Cot}\,10\rho\,\ldots)^{-1},$$

$$\sqrt[4]{b_2} = \{\text{Cot}\,.\,(2^2\rho)\;\text{Cot}\,.\,(2^2\,.\,3\rho)\,.\,\text{Cot}\,(2^2\,.\,5\rho)\,\ldots\}^{-1},$$

the further change of b_2 to b_3, changes 2^2 to 2^3 and so on. Therefore collectively

$$\sqrt{(bC)} = (\text{Cot}\,\rho\,.\,\text{Cot}\,2\rho\,.\,\text{Cot}\,3\rho\,.\,\text{Cot}\,4\rho\,\ldots)^{-1}.$$

Multiply this by $\sqrt{\dfrac{1}{b}} = (\text{Cot}\,\rho\,.\,\text{Cot}\,3\rho\,.\,\text{Cot}\,5\rho\,\ldots)^2,$

and you obtain $\sqrt{C} = \dfrac{\text{Cot}\,\rho}{\text{Cot}\,2\rho}\,.\,\dfrac{\text{Cot}\,3\rho}{\text{Cot}\,4\rho}\,.\,\dfrac{\text{Cot}\,5\rho}{\text{Cot}\,.\,6\rho}\ldots\,\&\text{c.},$

or $\tfrac{1}{2}\log C = l\,\text{Cot}\,\rho - l\,\text{Cot}\,2\rho + l\,\text{Cot}\,3\rho + \&\text{c.}$ as before.

This is mere corroboration: but in argument so various, corroboration is acceptable.]

Thus $\frac{1}{2}\log C$ is found by two very diverse series, each converging excellently.

Whether the series involving C^2 are of use, is not very clear. We have

$$\tfrac{1}{8}C^2 = \text{ק}\,(2\rho) + 3\text{ק}\,(6\rho) + 5\text{ק}\,(10\rho) + \&\text{c.}$$

$$\tfrac{1}{2}C_2 c = \text{ק}\,(\rho) + 3\text{ק}\,(3) + 5\,(5\rho) + \&\text{c.}$$

$$\tfrac{1}{2}C^2 cb = \text{Sec}\,\rho - 3\,\text{Sec}\,.\,3\rho + 5\,\text{Sec}\,5\rho - \&\text{c.}$$

To find $\aleph_c$ from the following, we need to know C^2; viz. from

$$\tfrac{1}{4}\,.\,C^2(1 - \aleph_n) = \text{ק}\,(2\rho) + 2\text{ק}\,(4\rho) + 3\text{ק}\,(6\rho) + \&\text{c.}$$

whence $$2(1 - \aleph_c) = \frac{\text{ק}\,(2\rho) + 2\text{ק}\,(4\rho) + 3\text{ק}\,(6\rho) + \&\text{c.}}{\text{ק}\,(2\rho) + 3\text{ק}\,(6\rho) + 5\text{ק}\,(10\rho) + \&\text{c.}}$$

$$= 1 + \frac{2\text{ק}\,(2\rho) + 4\text{ק}\,(4\rho) + 6\text{ק}\,(6\rho) + \&\text{c.}}{\text{ק}\,(2\rho) + 3\text{ק}\,(6\rho) + 5\text{ק}\,(10\rho) + \&\text{c.}}\,.$$

13. We mark as duplicate two different series for

$$\left.\begin{aligned}\tfrac{1}{2}\left(\log\frac{4}{c}-\rho\right)&=\text{ח}(\rho)-\tfrac{1}{2}\text{ח}(2\rho)+\tfrac{1}{3}\text{ח}(3\rho)-\&c.\\ \text{also}\qquad &=\log C-\tfrac{1}{2}\log C_1-\tfrac{1}{4}\log C_2-\tfrac{1}{8}\log C_3-\&c.\end{aligned}\right\}$$

$$\left.\begin{aligned}\tfrac{1}{4}\log\frac{1}{b}&=\text{ק}(2\rho)+\tfrac{1}{3}\text{ק}(6\rho)+\tfrac{1}{5}\text{ק}(10\rho)+\&c.\\ &=l\,\text{Cot}\,\rho+\text{Cot}\,3\rho+\text{Cot}\,5\rho+\&c.\end{aligned}\right\},$$

$$\left.\begin{aligned}\tfrac{1}{2}\log C&=\text{ח}\rho+\tfrac{1}{2}\text{ח}(2\rho)+\tfrac{1}{3}\text{ח}(3\rho)+\&c.\\ &=l\,\text{Cot}\,\rho-l\,\text{Cot}\,2\rho+l\,\text{Cot}\,3\rho-l\,\text{Cot}\,4\rho+\&c.\end{aligned}\right\},$$

$$\left.\begin{aligned}\tfrac{1}{2}(C-1)&=\text{כ}(\rho)-\text{כ}(3\rho)+\text{כ}(5\rho)-\text{כ}(7\rho)+\&c\\ &=\text{Sec}\,2\rho+\text{Sec}\,4\rho+\text{Sec}\,6\rho+\&c.\end{aligned}\right\},$$

$$\left.\begin{aligned}\tfrac{1}{2}Cc&=\text{ק}(\rho)-\text{ק}(3\rho)+\text{ק}(5\rho)-\&c.\\ &=\text{Sec}\,q+\text{Sec}\,3\rho+\text{Sec}\,5\rho+\&c.\end{aligned}\right\},$$

$$\left.\begin{aligned}\tfrac{1}{2}(1-bC)&=\text{ח}(\rho)-\text{ח}(3\rho)+\text{ח}(5\rho)-\&c.\\ &=\text{Sec}\,2\rho-\text{Sec}\,4\rho+\text{Sec}\,6\rho-\&c.\end{aligned}\right\}.$$

In No. 12 of Art. 10, make $\omega=\frac{1}{2}\pi=x$;

$$\therefore\ \tfrac{1}{2}\daleth_c \text{ or } \tfrac{1}{4}\log\frac{1}{b}=\text{ק}(2\rho)+\tfrac{1}{3}\text{ק}(6\rho)+\tfrac{1}{5}\text{ק}(10\rho)+\&c$$

as before from No. 15, a mere corroboration, yet not less welcome.

By a limited number of Tables easily computed, we can express, with ρ alone as basis, first $\text{ק}(\rho)$, Sec ρ, $\text{כ}(\rho)$ and $\text{ח}(\rho)$, next C, b, c $\left(\text{or } \log\frac{1}{c}\ ?\right)$ without difficulty. What is the *best* mode of computing $\aleph$ is not to me clear. A table for Cc is easily made, and *for rough purposes* the simple c may be inferred from $\frac{Cc}{C}$. For instance, if we wish to judge whether a closer value of the Grade γ is obtainable through $b=\cos\gamma$, or through $c=\sin\gamma$ when ρ is given, we ask first for a rough idea of the limits. To make this clearer I exhibit an outline of the value of c estimated from Cc divided by C.

ρ	c	ρ	c
1	·9440 8504	5	02694689
1·5	·7418 3940	6	00991476
2	·5029 9981	7	00364751
3	·1971 1112	8	...($C=1$
4	·0731 6436	9	...to 8 decimals

14. That which has been called the Problem of Section loses its interest now by its extreme ease. Given $F(c\omega)$, c and ω, to find θ when $F(c, \theta) = \frac{1}{n} . F(c\omega)$. Put $F(c\omega) = Cx$, then we account C and x known. Also put $F(c\theta) = Cy$, and y is not yet known. But

$$Cy = \frac{1}{n} . Cx, \therefore y = \frac{x}{n}.$$

We find θ from equation 11 of Art. 10 above in terms of y and ρ.

As a curiosity, we may transform

$$\tfrac{1}{4} C^2 (1 - \aleph_c) = \text{ק}(2\rho) + 2\text{ק}(4\rho) + 3\text{ק}(6\rho) + \&c. \text{ (a)},$$

$$= \frac{2q}{1-q^2} + \frac{4q^2}{1-q^4} + \frac{6q^3}{1-q^3} + \&c., \text{ as follows.}$$

Halve the equation, expand each term separately on the right,

$$\left.\begin{aligned} \tfrac{1}{8} C^2 (1 - \aleph_c) = \quad & q + q^3 + q^5 + q^7 + \&c. \\ & + 2(q^2 + q^6 + q^{10} + q^{14} + \&c.) \\ & + 3(q^3 + q^9 + q^{15} + q^{21} + \&c.) \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} .$$

Add up vertically by the formula

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \&c.$$

The total then is

$$\frac{q}{(1-q)^2} + \frac{q^3}{(1-q^3)^2} + \frac{q^5}{(1-q^5)^2} + \&c.$$

Multiply by 4,

$$\therefore \tfrac{1}{2} C^2 (1 - \aleph_c) = \left(\frac{2\sqrt{q}}{1-q}\right)^2 + \left(\frac{2\sqrt{q^3}}{1+q^3}\right)^2 + \left(\frac{2\sqrt{q^5}}{1-q^4}\right)^2 + \&c.$$

$$= (\text{ק}\rho)^2 + (\text{ק} . 3\rho)^2 + (\text{ק} . 5\rho)^2 + \&c. \text{ (b)}.$$

This a purely algebraic identity; $2(a) = (b)$.

15. With much variety and toil we may seem to have merely evaded the task in hand. Instead of evaluating F, E (and in the distance Π), we have shewn how elegantly the functions of ω, the amplitude of F, E, are related to x, an average value of F; which *average* is not very easy to fix with high accuracy, even by help of Legendre's table of *double entry*. Just so, if Trigonometry had not preceded the Integral Calculus, the investigation of

$$\int_0 \frac{dx}{\sqrt{(1-x^2)}} \text{ and } \int_0 \frac{dx}{1+x^2}$$

would have led through $\sin^{-1}x$ and $\tan^{-1}x$ to the demand of Trigonometrical Tables with a new argument, the arc ω, making either $x = \sin\omega$, or $x = \tan\omega$. But here the constant element c gives a new complexity, and its relation to other constants more convenient requires new Tables.

Here it is a great simplification to pass from the original c to ρ, and whenever actual evaluation is required, our first need may seem to be Tables for the transformation.

If a man expert in this part of the Calculus be asked, "What is the simplest *direct* method of computing x when c and ω are given?" he must reply; By Lagrange's Scale, from ω he will find in succession $\omega_1, \omega_2, \omega_3 \ldots$ and perhaps ω_4 will suffice. Each new ω is deduced from its predecessor by an equation of like type: namely

$\tan(\omega_1 - \omega) = b\tan\omega$ enables him to fix the degrees in ω_1;

from $\tan(\omega_2 - \omega_1) = b_1 \tan\omega_1$ he deduces ω_2;

from $\tan(\omega_3 - \omega_2) = b_2 \tan\omega_2$ he deduces ω_3; and so on.

But this supposes that from c he has found numerically and accurately $b, b_1, b_2, b_3 \ldots$ *or rather their logarithms*, for he will work by the form

$$\log\tan(\omega_1 - \omega) = \log\tan(\omega) - \log\frac{1}{b};$$

Legendre's tables excellently furnish for every given arc γ (with $c = \sin\gamma$, $b = \cos\gamma$) the values of $\log b$, $\log b_1$, $\log b_2$, $\log b_3 \ldots$ but even so, to be accurate to 10 decimals for $\omega_1, \omega_2, \omega_3, \omega_4 \ldots$ is not a task for even high mathematicians. Legendre himself rejoiced in his own scale, *because*, by its higher convergence, it promised higher accuracy without greater toil. Yet alas! no one can practically use this new scale, from the difficulty of its constants.

16. Yet if once we resolve to make ρ our chief constant instead of c—which can be effected by a table of single entry, as easily as to pass from an arc γ to $\log\tan\gamma$ &c.—the constants needed in Legendre's scale are as easy as those in Lagrange's scale. Not knowing whether this treatise would ever be published, I anticipated this statement in a recent short treatise, called Anticyclics. It seems right here to fill out the argument.

In this scale c_1 is connected to c, as 3ρ to ρ, and our typical equation is $\tan\frac{1}{2}(\omega_1 - \omega) = \tan\omega \,.\, \Delta(c, \beta)$, where β is defined by the equation

$F(c,\beta)=\frac{2}{3}F(c,\frac{1}{2}\pi)$, which at once shows that x' (Mesonome of β) $=\frac{1}{3}\pi$. Hence equation 15 of Art. 10 above, gives

$$-\tfrac{1}{2}\log\Delta\,(c,\beta)=\frac{1-\cos 2x'}{\operatorname{Sin}2\rho}+\frac{1-\cos 6x'}{3\operatorname{Sin}6\rho}+\frac{1-\cos 10x'}{5\operatorname{Sin}10\rho}+\&c.$$

Now when n is a multiple of 3, say, $=3m$, in the general term of the last;

$$\cos 2nx'=\cos\tfrac{2}{3}n\pi=\cos 2m\pi=1\ \ (m\text{ being integer}),$$

therefore the term vanishes. But when n has the form $3m\pm 1$,

$$\cos 2nx'=\cos\,(3m\pm 1)\,\tfrac{2}{3}\pi=\cos\,(2m\pi\pm 120^{\circ})=-\tfrac{1}{2}.$$

This at once yields $\frac{3}{2}$ as the numerator of every such term. Finally

$$-\tfrac{1}{3}\log\Delta\,(c,\beta)=\text{ק}(2\rho)+\tfrac{1}{5}\text{ק}(10\rho)+\tfrac{1}{7}\text{ק}(14\rho)+\tfrac{1}{11}\text{ק}(22\rho)\ldots$$

with enormous convergence. By this a table is easily computed. But it must be multiplied by M (the modulus) to convert the result to *common* logarithms.

For a moment let $-M.\log_e\Delta\,(c\beta)$ or $-\log_{10}.\,\Delta\,(c\beta)$ be denoted by D; and let $D, D_1, D_2\ldots$ in this scale answer to $c, c_1, c_2\ldots$ i.e. to $\rho\ 3\rho\ 3^2\rho\ 3^3\rho\ldots$ Then with

$$\log_{10}\tan\tfrac{1}{2}(\omega_1-\omega)=\log_{10}\tan\omega-D,$$

which means

$$D=3M\{\text{ק}(2\rho)+\tfrac{1}{5}\text{ק}(10\rho)+\tfrac{1}{7}\text{ק}(14\rho)+\tfrac{1}{11}\text{ק}(22\rho)+\&c.\},$$

whence also

$$D_1=3M\{\text{ק}(6\rho)\ +\tfrac{1}{5}\text{ק}(30\rho)+\tfrac{1}{7}\text{ק}(42\rho)\ldots\}$$

$$D_2=3M\{\text{ק}(18\rho)+\tfrac{1}{5}\text{ק}(90\rho)+\ldots\}.$$

In D_2 the first term is ample. Where ρ is as large as 12·6, the difference of $\text{ק}(\rho)$ and $2\epsilon^{-\rho}$ does not affect the 16th decimal; and when $\rho=2$ (Gudemann's lowest limit) $3M.\,\text{ק}(18\rho)$ probably begins with 15 zeros.

For illustration merely, I here reprint my table of

$$-\log_{10}.\sqrt{(1-c^2\sin^2\beta)};$$

where $\frac{1}{3}\pi$ is Mesonome of β.

17. **Further concerning the constants of Legendre's scale.** When c^2 is given, we have to solve a biquadratic equation

$$c^2 = \frac{(3\mu - 1)(1+\mu)^3}{16\mu^3},$$

to find his Regulator μ. But when we start from ρ as known, our equation

$$\log \tan . \tfrac{1}{2}\omega = \log \tan . \tfrac{1}{2}x$$
$$+ 2\{\text{ח}(\rho) \cos x + \tfrac{1}{3}\text{ח}(3\rho) \cos 3x + \tfrac{1}{5}\text{ח}(5\rho)\cos 5x + \ldots\},$$

allows us to put $\omega = \beta$, $x = \frac{1}{3}\pi$, and we know that

$$\mu = \frac{1 - \cos\beta}{1 + \cos\beta}, \quad \therefore \sqrt{\mu} = \tan\tfrac{1}{2}\beta.$$

In this series make $\omega = \beta$, $\frac{1}{2}x = \frac{1}{6}\pi$ or 30°, $\tan 30^\circ = \frac{1}{\sqrt{3}}$, $x = 60^\circ$,

$$\cos x = \tfrac{1}{2}, \; \cos 3x = -\tfrac{1}{2}, \; \cos 5x = -\tfrac{1}{2}, \; \cos 7x = +\tfrac{1}{2};$$

$$\therefore \tfrac{1}{2}\log\mu = -\tfrac{1}{2}\log 3 + 2\{\text{ח}(\rho)\tfrac{1}{2} - \tfrac{1}{3}\text{ח}(3\rho)\tfrac{1}{2} - \tfrac{1}{5}\text{ח}(5\rho)\tfrac{1}{2} + \&c.\},$$

$$\tfrac{1}{2}\log(3\mu) = \text{ח}(\rho) - \tfrac{1}{3}\text{ח}(3\rho) - \tfrac{1}{5}\text{ח}(5\rho) + \tfrac{1}{7}\text{ח}(7\rho) + \tfrac{1}{9}\text{ח}(9\rho) - \&c.,$$

or again, we may use the more general formula, $C = 3\mu C_1$ [which, it will appear, in the scale of Index $= n$, becomes $C = n\mu C_1$]. But we know C as a function of ρ; change ρ to 3ρ, and you find C_1. In fact $\log(3\mu) = \log C - \log C_1$ which is peculiarly easy, if we have a table of $\log C$ with argument ρ.

In my Anticylics, the table of $\text{ל}(\rho)$ gives $\frac{1}{2}\log\frac{1}{b}$; the table of $\text{ט}(\rho) = \frac{1}{2}\log C$; that of $\text{ר}(\rho) = \frac{1}{2}(C-1)$, and that of $\text{ז}(\rho) = \log_e Q$.

I have not been able to attempt a table of $\aleph_c$ from ρ as argument.

The β is defined by $F(c\,\beta) = \frac{2}{3} F_c$.

ρ	$-\log_{10}\sqrt{(1 - c^2 \sin^2\beta)}$	ρ	$-\log_{10}\sqrt{(1 - c^2 \sin^2\beta)}$
1·0	·3592 5572 9411 6719	4·0	8 7413 7507 5280
1·1	·2923 2484 2664 4562	4·1	7 1568 3233 2307
1·2	·2383 5463 1860 7711	4·2	5 8595 1857 9358
1·3	·1946 1441 0045 9939	4·3	4 7973 6797 9182
1·4	·1590 4540 8993 1846	4·4	3 9277 5265 4496
1·5	·1300 5603 0350 5642	4·5	3 2157 7186 4632
1·6	·1063 9363 9293 4314	4·6	2 6318 5130 7237
1·7	·0870 5994 8696 7943	4·7	2 1555 9632 6260
1·8	·0712 5245 5310 8973	4·8	1 7648 5299 9547
1·9	·0583 2220 7250 4299	4·9	1 4449 3942 3209
2·0	·0477 4230 2244 3056	5·0	1 1830 1634 0904
2·1	·0390 8376 6653 7070	5·1	9685 7005 9051
2·2	·0319 9670 8941 8197	5·2	7929 9956 7204
2·3	·0261 9538 7599 8236	5·3	6492 5313 2650
2·4	0214 4625 4932 5163	5·4	5315 6350 6124
2·5	·0175 5651 6359 9217	5·5	4352 0738 9615
2·6	·0142 8531 8618 4305	5·6	3563 1767 3812
2·7	·0117 6939 6969 2738	5·7	2917 2823 7398
2·8	·0096 3590 2604 9745	5·8	2388 4687 9488
2·9	·0078 8917 4230 4561	5·9	1955 5128 5509
3·0	0064 6209 0040 0540	6·0	1601 0385 1246
3·1	52 8824 4940 6155	6·1	1301 8194 6701
3·2	43 2964 2883 7997	6·2	1073 2082 0936
3·3	35 5301 5415 2412	6·3	878 6685 6544
3·4	29 0224 2004 9576	6·4	719 3926 7729
3·5	22 4736 5465 9947		
3·6	19 4542 9665 4372		
3·7	17 2307 1146 9322		
3·8	13 0406 0102 6271		
3·9	10 6767 4021 7259		

We now can extend the table as far as we please (true to 16 decimals) by the formula

$$6M \,.\, \epsilon^{-2\rho},$$

if the factor M is the modulus of the *common logarithms*.

From Legendre.

Let $c = \sin\gamma$ and $b = \cos\gamma$.

$\gamma =$ the grade.	$F_c = \frac{1}{2}\pi C$.	$F_b = \frac{1}{2}\pi B$.
0°	1·5707 9632 6795	log ·00
1	1·5709 1595 8127	5·4349 0982 9625
2	1·5712 7495 2372	4·7427 1726 5279
3	1·5718 7361 0514	4·3386 5397 6000
4	1·5727 1243 4995	4·0527 5816 9549
5	1·5737 9213 0925	3·8317 4199 9766
6	1·5751 1360 7777	3 6518 5596 9479
7	1·5766 7798 1593	3·5004 2249 9173
8	1·5784 8657 7689	3·3698 6802 6668
9	1·5805 4093 3896	3·2553 0294 2143
10	1·5828 4280 4338	3·1533 8525 1888
11	1·5853 9416 3775	3·0617 2861 2039
12	1·5881 9721 2527	2·9785 6895 1181
13	1·5912 5438 2014	2·9025 6494 0670
14	1·5945 6834 0932	2·8326 7258 2918
15	1·5981 4200 2113	2·7680 6314 5369
16	1·6019 7853 0087	2·7080 6761 4590
17	1·6060 8134 9410	2·6521 3800 4630
18	1·6014 5415 3790	2·5998 1973 0061
19	1·6151 0091 6068	2·5507 3144 9627
20	1·6200 2580 9124	2·5045 5007 9002
21	1·6252 3366 7759	2·4609 9945 8304
22	1·6307 2910 1631	2·4198 4165 3739
23	1·6365 1740 9336	2·3808 7019 0604
24	1·6426 0414 3713	2·3439 0472 4447

$c = \sin \gamma$. $b = \cos \gamma$.

γ	F_c	F_b
25°	1·6489 9522 8479	2·3087 8679 8167
26	1·6556 9692 6310	2·2753 7642 9612
27	1·6627 1595 8491	2·2435 4934 1699
28	1·6700 5942 6270	2·2131 9469 4981
29	1·6777 3488 4081	2·1842 1321 6949
30	1·6857 5035 4813	2·1565 1564 7500
31	1·6941 1435 7306	2·1300 2143 8399
32	1·7028 3523 6412	2·1046 5765 8491
33	1·7119 2469 5156	2·0803 5806 6692
34	1·7213 9083 1374	2·0570 6232 2797
35	1·7312 4517 5657	2·0347 1531 2186
36	1·7414 9923 4427	2·0132 6656 5201
37	1·7521 6523 6469	1·9926 6975 5735
38	1·7632 5618 4059	1·9728 8226 6275
39	1·7747 8590 9104	1·9538 6480 9252
40	1·7867 6913 4885	1·9355 8109 6006
41	1·7992 2154 4050	1·9179 9754 6438
42	1·8121 5985 3662	1·9010 8303 3465
43	1·8256 0189 8136	1·8848 0865 7384
44	1·8395 6672 1094	1·8691 4754 6031
45	1·8540 7467 7301	1·8540 7467 7301

New use of Λ *and* Θ *in factors.*

18. Hitherto our argument has used Λ and Θ as expanded in their first definition; but as deduced from S and T, they stand as products of factors.

Their general factor is $(1-2q^n \cos 2x + q^{2n})$, which may be cast into three forms, which, when q is constant and only x varies, are

$(1-q^n)^2 + 4q^n \sin^2 x$; $(1+q^n)^2 - 4q^n \cos^2 x$; $(1+q^{2n}) - 2q^n . \cos 2x$: proportional to

$$1 + \frac{\sin^2 x}{\operatorname{Sin}^2 n\rho}; \quad 1 - \frac{\cos^2 x}{\operatorname{Cos}^2 . n\rho}; \quad 1 - \frac{\cos 2x}{\operatorname{Cos} 2n\rho}.$$

Therefore with $A_1 A_2 A_3$ unknown functions of q, we may assume three equations

$$\left.\begin{aligned}
\frac{\Lambda(q, x)}{A_1} &= \sin x . \left(1 + \frac{\sin^2 x}{\operatorname{Sin}^2 . 2\rho}\right)\left(1 + \frac{\sin^2 x}{\operatorname{Sin}^2 . 4\rho}\right)\left(1 + \frac{\sin^2 x}{\operatorname{Sin}^2 . 6\rho}\right) \ldots \&c.\\
\frac{\Lambda(q, x)}{A_2} &= \sin x . \left(1 - \frac{\cos^2 x}{\operatorname{Cos}^2 . 2\rho}\right)\left(1 - \frac{\cos^2 x}{\operatorname{Cos}^2 . 4\rho}\right)\left(1 - \frac{\cos^2 x}{\operatorname{Cos}^2 . 6\rho}\right) \ldots \&c.\\
\frac{\Lambda(q, x)}{A_3} &= \frac{\sin x}{\sin \frac{1}{4}\pi} . \left(1 - \frac{\cos 2x}{\operatorname{Cos} 4\rho}\right)\left(1 - \frac{\cos 2x}{\operatorname{Cos} 8\rho}\right)\left(1 - \frac{\cos 2x}{\operatorname{Cos} 12\rho}\right) \ldots \&c.
\end{aligned}\right]$$

To find A_1 make $x = 0$; then $A_1 = \frac{\Lambda(q, x)}{\sin x} = \frac{\Lambda(q, x)}{x} = \Lambda'(0)$. [See (j) in Ch. VII. 15.]

To find A_2 make $x = \frac{1}{2}\pi$; $\therefore A_2 = \Lambda(\frac{1}{2}\pi)$.

To find A_3 make $x = \frac{1}{4}\pi$; $A_3 = \Lambda(\frac{1}{4}\pi)$.

Similarly with $B' B'' B'''$ unknown constants, assume

$$\left.\begin{aligned}
\frac{\Theta(q, x)}{B'} &= \left(1 + \frac{\sin^2 x}{\operatorname{Sin}^2 . \rho}\right)\left(1 + \frac{\sin^2 x}{\operatorname{Sin}^2 . 3\rho}\right)\left(1 + \frac{\sin^2 x}{\operatorname{Sin}^2 . 5\rho}\right) \ldots \&c.\\
\frac{\Theta(q, x)}{B''} &= \left(1 - \frac{\cos^2 x}{\operatorname{Cos}^2 . \rho}\right)\left(1 - \frac{\cos^2 x}{\operatorname{Cos}^2 . 3\rho}\right)\left(1 - \frac{\cos^2 x}{\operatorname{Cos}^2 . 5\rho}\right) \ldots \&c.\\
\frac{\Theta(q, x)}{B'''} &= \left(1 - \frac{\cos 2x}{\operatorname{Cos} . 2\rho}\right)\left(1 - \frac{\cos 2x}{\operatorname{Cos} . 6\rho}\right)\left(1 - \frac{\cos 2x}{\operatorname{Cos} . 10\rho}\right) \ldots \&c.
\end{aligned}\right]$$

To find B', make $x = 0$; then $B' = \Theta(0)$.

To find B'', make $x = \frac{1}{2}\pi$; then $B'' = \Theta(\frac{1}{2}\pi)$.

To find B''', make $x = \frac{1}{4}\pi$; then $B''' = \Theta(\frac{1}{4}\pi)$.

19. We found $\quad \daleth(q, x) = \log \dfrac{\Theta(q, x)}{\Theta(q, 0)}$,

which, hence [since $\Theta(q, 0) = B'$]

$$= \Sigma \log \left(1 + \frac{\sin^2 x}{\mathrm{Sin}^2 . (2n-1)\rho}\right),$$

if n mean 1, 2, 3, 4 ....

Differentiate, remembering that

$$d . \daleth = G . dF; \quad dF = C dx;$$

also $\quad d \log \left(1 + \dfrac{\sin^2 x}{a}\right) = \dfrac{d . \sin^2 x}{a + \sin^2 x} = \dfrac{2 \sin 2x . dx}{2a + (1 - \cos 2x)}$;

where a means $\mathrm{Sin}^2 (2n-1)\rho$, therefore

$$C . G(c, \omega) = \Sigma . \frac{2 \sin 2x}{1 + 2\,\mathrm{Sin}^2 (2n-1)\rho - \cos 2x}.$$

A somewhat simpler series results more directly from

$$d . \daleth(q, x) = d \log \Theta(q, x) = \frac{d\Theta(q, x)}{\Theta(q, x)};$$

or $\quad \tfrac{1}{2} C . G(c\omega) = \dfrac{2q^{1.1} \sin 2x - 4q^{2.2} \sin 4x + 6q^{3.3} \sin 6x - \&c.}{1 - 2q^{1.1} \cos 2x + 2q^{2.2} \cos 4x - 2q^{3.3} \cos 6x + \&c.}$.

If we have an adequate table of $\epsilon^{-\rho}$, and ρ is given, we directly find the powers of $q = \epsilon^{-2\rho}$, $q^{n.n} = \epsilon^{-2n.n\rho}$. The convergence is unprecedented.

If occasion requires (though it never seems to require), that we need to calculate $E(c\omega)$, we must have recourse to the primary method of Ch. II., else deduce it from $G(c\omega)$. But the latter method needs $\aleph_c$, of which we have no table, and thus we are driven to Legendre's table of F_c and E_c, or rather of $\log F_c$ and $\log E_c$ for

$$\log \aleph_c = \log E_c - \log F_c.$$

Since Legendre undoubtedly calculated ρ as his step towards F_b, and $\aleph_c$ as his step towards E_c (though without my notation), it is always possible that his *subsidiary* work for full tables of ρ and $\aleph_c$ may be recovered.

Large Modulus.

20. If c be very near to 1, i.e. too *large*, q may be too small. Then let ρ' r be to b, what ρ q are to c.

I called q the Basis, let r be the Antibasis. Then as

$$\log\frac{1}{q} = 2\rho = \pi \cdot \frac{B}{C},$$

so

$$\log\frac{1}{r} = 2\rho' = \pi \cdot \frac{C}{B};$$

and

$$4\rho\rho' = \pi^2, \quad = \log\frac{1}{q} \cdot \log\frac{1}{r}.$$

When $b = c$, $q = r$. But $\log_e 23 =$ nearly π, and $q = \frac{1}{23} = r$, nearly.

If q exceeds $\frac{1}{23}$, r is less than $\frac{1}{23}$.

If q_1 is $= q^2$, and r_1 is related to q_1 as r to q

$$\log\left(\frac{1}{q_1}\right) \cdot \log\left(\frac{1}{r_1}\right) = \pi^2,$$

or

$$\log\frac{1}{q^2} \cdot \log\frac{1}{r} = \log\frac{1}{q} \cdot \log\frac{1}{r}.$$

But

$$\log\frac{1}{q^2} = 2\log\frac{1}{q}, \quad \therefore \log\frac{1}{r_1} = \frac{1}{2}\log\frac{1}{r},$$

or

$$r_1 = \sqrt{r}.$$

Thus a series $q\ q^2\ q^4\ q^8 \ldots$ answers to $r\ \sqrt{r}\ \sqrt[4]{r}\ \sqrt[8]{r} \ldots$.

COR. In this connection it is convenient sometimes to write a for $\frac{C}{B}$, and ξ for $\frac{x}{\pi}$,

$$2\rho' = a\pi, \quad \therefore r = \epsilon^{-2\rho'} = \epsilon^{-a\pi}, \quad r^{\xi} = r^{\frac{x}{\pi}} = \epsilon^{-ax}.$$

21. As with the Antinome in Ch. III., when

$$\sin\omega = \sqrt{-1} \cdot \tan\psi, \quad F(c\omega) = \sqrt{-1} \cdot F(b\psi),$$

or if x, y are the Mesonomes,

$$Cx = \sqrt{-1} \cdot By,$$

or

$$ax = \sqrt{-1} \cdot y, \quad -\frac{x^2}{2\rho} = \frac{y^2}{2\rho'}.$$

Also in Ch. III. Art. 15, we had the transformation

$$\daleth(c\omega) = \daleth(b\psi) - \frac{x^2}{2\rho} + \log\cos\psi.$$

Replace each $\daleth$ by our new form

$$\daleth(c\omega) = \log . \frac{\Theta(q, x)}{\Theta(q, 0)},$$

then $$\frac{\Theta(q, x)}{\Theta(q, 0)} = \frac{\Theta(r, y)}{\Theta(r, 0)} . \epsilon^{-\frac{x^2}{2\rho}} . \cos\psi.$$

Again, as $$\sqrt{c} . \cos\omega = \sqrt{b} . \frac{\Lambda^0}{\Theta},$$

so $$\sqrt{b}\cos\psi = \sqrt{c} . \frac{\Lambda^0(r, y)}{\Theta(r, y)}.$$

Eliminate $$\Theta(r, y)\cos\psi,$$

hence $$\frac{\Theta(q, x)}{\Theta(q, 0)} = \sqrt{\frac{c}{b}} . \frac{\Lambda^0(r, y)}{\Theta(r, 0)} . \epsilon^{-\frac{x^2}{2\rho}}.$$

Make $$x = 0, \quad y = 0,$$

$$\therefore \quad 1 = \sqrt{\frac{c}{b}} . \frac{\Lambda^0(r, 0)}{\Theta(r, 0)}.$$

Eliminate $$\sqrt{\frac{c}{b}},$$

then $$\frac{\Theta(q, x)}{\Theta(q, 0)} = \frac{\Lambda^0(r, y)}{\Lambda^0(r, 0)} . \epsilon^{-\frac{x^2}{2\rho}} \quad (1).$$

Such is Legendre's beautiful result, Second Supplement, Art. 182.

22. Further

$$\Theta(q, 0) = \sqrt{(bC)}, \quad \Lambda^0(r, 0) \text{ or } \Lambda(r, \tfrac{1}{2}\pi) = \sqrt{(bB)}, \quad \frac{C}{B} = a,$$

so that $$\frac{\Theta(q, 0)}{\Lambda^0(r, 0)} = \sqrt{\frac{bC}{bB}} = \sqrt{a}.$$

$$\therefore \quad \Theta(q, x) = \sqrt{a} . \Lambda^0(r, y)\, \epsilon^{-\frac{x^2}{2\rho}} \quad (2).$$

Solving in reverse,

$$\Lambda^0(r, y) = \sqrt{a^{-1}} . \Theta(q, x)\, \epsilon^{\frac{x^2}{2\rho}}.$$

In this, the change of $c\omega$ to $b\psi$ require the change of $b\psi$ to c and $-\omega$, observing that

$$ax=\sqrt{-1}\,y \text{ leads to } a^{-1}y=-x\sqrt{-1};$$

so x changes to $-y$, a to a^{-1}, ρ to ρ'.

Then $$\Lambda^0(q,\ x)=\sqrt{a}\,.\,\Theta(r,\ -y)\,\epsilon^{\frac{y^2}{2\rho}}$$

$$=\sqrt{a}\,.\,\Theta(r,\ y)\,\epsilon^{-\frac{x^2}{2\rho}} \text{ [see 21]};$$

and if ξ stands for the *ratio* $\frac{x}{\pi}$ (which is not an *arc*),

$$\Theta(q,\ x)=\sqrt{a}\Lambda^0\ (r,\ y)\,.\,r^{\xi^2} \quad (3),$$

$$\Lambda^0(q,\ x)=\sqrt{a}\Theta\ (r,\ y)\ \ r^{\xi^2} \quad (4).$$

23. Now by definition of Λ, and hence of $\Lambda(r,\ y+\frac{1}{2}\pi)$

$$\Lambda^0(r,\ y)=\Sigma\,\{r^{(m-\frac{1}{2})^2}\,.\,2\cos(2m-1)\,y\}$$

where m means 1, 2, 3, 4 ....

Transform by $y=-\sqrt{-1}\,.\,ax$; $2\cos.\,ny=2\operatorname{Cos}(nax)$.

Here, the circular arc changing to an Anticyclic, we may well feel anxiety about the convergence.

But, whatever n,

$$2\operatorname{Cos}(nax)=\epsilon^{nax}+\epsilon^{-nax}=r^{-n\xi}+r^{n\xi},$$

and in $$\Sigma\,\{r^{\frac{1}{4}n^2}\,.\,2\operatorname{Cos}.\,nax\}$$

the exponent of r is $$\tfrac{1}{4}n^2\mp n\xi;$$

but to attain $\Theta(q,x)$ by (3), we have further to multiply by r^{ξ^2} which makes the total exponent of r

$$=\tfrac{1}{4}n^2\mp n\xi+\xi^2 \quad\text{or}\quad (\tfrac{1}{2}n\mp\xi)^2,$$

so that $$\Theta(q,\ x)=\sqrt{a}\,.\,\Sigma r^{(\frac{1}{2}n\mp\xi)^2}$$

if we give to n its right successive values, which are found from $2m-1=n$; where m meant 1, 2, 3, 4 ... or $n=1, 3, 5, 7$.

Further, let $x=(\frac{1}{2}-u)\,\pi$; then as we may suppose x less than π, u is positive, but less than $\frac{1}{2}$, and

$$\xi=\frac{x}{\pi}=\frac{1}{2}-u.$$

In $(\frac{1}{2}n \mp \xi)^2$ take the double sign separately, therefore

$$(\tfrac{1}{2}n - \xi) \quad \text{or} \quad \frac{n-1}{2} + u$$

means the series $\quad u, \ 1+u, \ 2+u, \ \&c. \ldots,$

and $$(\tfrac{1}{2}n + \xi) \quad \text{or} \quad \frac{n+1}{2} - u$$

means the series $\ 1-u, \ 2-u, \ 3-u \ldots$ all positive,
but that is unimportant, for here only the squares enter the exponent. Thus each series is contained in

$$u, \quad u \pm 1, \quad u \pm 2, \quad u \pm 3, \ldots$$

that is, simply in $(u+n)$ if n is understood for $0, \pm 1, \pm 2, \pm 3 \ldots$ including both signs. Then simply

$$\Theta(q, x) = \sqrt{a} \,.\, \Sigma r^{(u+n)} \qquad (5).$$

Since by hypothesis c is inconveniently large, ρ is small, ρ' large, r small; also u is less than $\frac{1}{2}$. Thus with the increase of x and n, the convergence is excellent.

That both Cyclics and Anticyclics here vanish, caused me at first much surprize; but it is correct.

24. Similarly to transform $\Lambda^0(q, x)$ by aid of equation (4), we must follow step by step in the last Article, *mutatis mutandis.* Only to prevent confusion of plus and minus, in the result as well as in the process, I *now* begin by $x' = \frac{1}{2}\pi - x$, which changes $\Lambda^0(q, x)$ to $\Lambda(q, x')$, next write $v = \dfrac{x'}{\pi}$. But from $x = (\frac{1}{2} - u)\pi$, and $\frac{1}{2}\pi - x = u\pi$, you have exactly $v = u$.

Because of the alternate signs, it is safer now to write at full

$$\begin{aligned}\Lambda^0(q, x) = \sqrt{a}\,\{r^{v^2} &- (r^{(1-v)^2} + r^{(1+v)^2}) \\ &+ (r^{(2-v)^2} + r^{(2+v)^2}) \\ &- (r^{(3-v)^2} + r^{(3+v)^2}) \\ &+ (r^{(4-v)^2} + r^{(4+v)^2}) - \&c.\} \qquad (6).\end{aligned}$$

Now we see how the vanished Anticyclics will appear in (5), and why they are no longer formidable.

25. To apply this to the Diplonome, where

$$\daleth(c,\ \omega) = \log \frac{\Theta(q,\ x)}{\Theta(q,\ 0)};$$

we have

$$\Theta(q,\ 0) = \sqrt{(bC)}, \quad a = \frac{C}{B}, \quad \therefore \frac{\sqrt{a}}{\Theta(q,\ 0)} = \sqrt{\frac{1}{bB}},$$

whence

$$\frac{\Theta(q,\ x)}{\Theta(q,\ 0)} = \sqrt{\frac{1}{bB}} \cdot \Sigma r^{(u+n)^2} n; \text{ as in (5) of 23,}$$

$$\therefore \quad \daleth(c\omega) = -\tfrac{1}{2}\log(bB) + \log \Sigma r^{(n+u)^2} \quad (7).$$

26. Next from $\daleth(c\omega) = \log \dfrac{\Theta(q,\ x)}{\Theta(q,\ 0)}$; we get

$$C \,.\, G(c\omega) = \frac{d\,.\,\Theta(q,\ x)}{\Theta(q, x)\,.\,dx},$$

since $$d\,.\daleth(c\omega) = C\,.\,G(c\omega).$$

But $$dF(c\omega) = \frac{d\omega}{\Delta(c\omega)} = C\,dx,$$

and we have made $\dfrac{x}{\pi} = \tfrac{1}{2} - u$, or $dx = -\pi u$.

$$\frac{d\,.\,\Theta(q,\ x)}{\Theta(q,\ 0)} \text{ in last Article} = \frac{d\Sigma\,.\,r^{(n+u)^2}}{\sqrt{(bB)}},$$

and we have to differentiate the general term $r^{(n+u)^2}$.

Whatever z may be, $d\,.\,r^z = \log r\,.\,r^z dz$. Make $z = (n+u)^2$.

Now $$\log r = -\log\frac{1}{r} = -\pi\,.\,\frac{C}{B}, \quad du = -\frac{dx}{\pi};$$

which two negatives are to be multiplied together as factors under Σ with result

$$d\,.\,r^{(n+u)^2} = +\pi\,.\,\frac{C}{B}\,.\,r^{(n+u)^2}\,.\,2(n+u)\,.\,\frac{dx}{\pi}.$$

The factor π in numerator and denominator disappears. The factor $\dfrac{C}{B}$ will change the factor C on the left to B; and

$$B\,.\,G(c\omega) = \frac{\Sigma\,.\,2(n+u)\,r^{(n+u)^2}}{\Sigma\left(r^{(n+u)^2}\right)} = 2u + \frac{\Sigma\,.\,2n\,.\,r^{(n+u)^2}}{\Sigma\left(r^{(n+u)^2}\right)},$$

n being, as before, the double series $0 \pm 1 \pm 2 \pm 3 \ldots.$ Observe, that since $2u = 1 - \frac{2x}{\pi}$, we may transpose $-\frac{2x}{\pi}$, and on the left, for $\frac{2x}{\pi}$ write $B\left(\frac{2x}{\pi B}\right)$, then $G(c\omega) + \frac{2x}{\pi B} = J(c\omega)$.

Thus for a large modulus c, we find (as might have perhaps been expected)

$$B \,.\, J(c\omega) = 1 + \frac{\Sigma \,.\, 2n \,.\, r^{(n+u)^2}}{\Sigma \,.\, r^{(n+u)^2}}.$$

27. It remains to inquire into the *factors* of Θ and Λ, akin to those in VII. 10, when r is to supersede q, because c is too large. The constant $\phi(q)$ was a factor, when $1 - 2q^m \cos 2x + q^{2m}$ was the general factor, and the latter arose from combining two imaginary factors of S and T. It will naturally relapse into two factors, when $\cos 2x$ changes into an Anticyclic. Here, to apply (3) and (4) of Art. 22, we must express $\Lambda^0(ry)$ [and presently $\Theta(ry)$ by the formula of VII. 10, using $\phi(r)$ instead of $\phi(q)$ of that Article. First

$$\phi(r) \,.\, \Lambda^0(r, y) = 2\cos y \,.\, r^{\frac{1}{4}}(1 + 2r^2 \cos 2y + r^4)$$
$$(1 + 2r^4 \cos 2y + r^8)(1 + 2r^6 \,\&\text{c.} \ldots).$$

Now $\quad ax = \sqrt{-1} \,.\, y, \; 2\cos y = 2\,\text{Cos}\,.\, ax, \; 2\cos 2y = 2\,\text{Cos}\, 2ax.$

But $$r = \epsilon^{-a\pi}, \; \xi = \frac{x}{\pi}, \; r^{\xi} = \epsilon^{-ax},$$

$$2\cos y = r^{-\xi} + r^{\xi}; \; 2\cos 2y = r^{-2\xi} + r^{2\xi},$$

$\therefore$ as general factor $1 + 2r^n \cos 2y + r^{2m} = 1 + r^m \,.\, (r^{-2\xi} + r^{2\xi}) + r^{2x}]$
$$= (1 + r^{m-2\xi})(1 + r^{2n+2\xi}),$$

the factor, as we foresaw, splitting into two.

Here m means 2, 4, 6, 8....

Writing a P of German type for Product, we express $\phi(r)\Lambda^0(ry)$. Then, multiply (3) of Art. 22, by $\phi(r)$, and you find

$$\phi(r) \,.\, \Theta(q, x) = \sqrt{a}, r^{\xi^2} \,.\, \{r^{\frac{1}{4}} \,.\, (r^{-\xi} + r^{\xi}) \,\mathfrak{P}\, (1 + r^{m-2\xi})(1 + r^{m+2\xi})\},$$

in which $\quad r^{\xi^2} \,.\, r^{\frac{1}{4}} \,.\, (r^{-\xi} + r^{\xi}) = r^{(\xi-\frac{1}{2})^2} + r^{(\xi+\frac{1}{2})^2},$

and m has the value 2, 4, 6, 8....

We had $\xi = \frac{x}{\pi} = \frac{1}{2} - u, \; \xi - \frac{1}{2} = -u, \; \xi + \frac{1}{2} = 1 - u, \; 2\xi = 1 - 2u.$

The series $2 - 2\xi, \; 4 - 2\xi, \; 6 - 2\xi, \ldots$ is $1 + 2u, \; 3 + 2u, \; 5 + 2u.$

The series $2+2\xi,\ 4+2\xi,\ 6+2\xi$ is $3-2u,\ 5-2u,\ 7-2u\ldots.$

Also $\qquad r^{(\xi-\frac{1}{2})^2}+r^{(\xi+\frac{1}{2})^2}=r^{(\frac{1}{2}-\xi)^2}(1+r^{2\xi})=r^{u^2}.(1+r^{1-2u}),$

so that after the factor r^{u^2} we have factors of the form $1+r^{n\pm 2u}$, in which r has exponents $1+2u,\ (3+2u)(5+u),$ and $1-2u,\ 3-2u,\ 5-2u,\ldots$ from values of n.

Finally, $\qquad \phi(r).\Theta(q,x)=\sqrt{a}.r^{u^2}.\mathfrak{P}(1+r^{n\pm 2u}).$

N.B. $\phi(r)=\dfrac{2r^{\frac{1}{4}}}{\sqrt{(bc.B^8)}}$: with n meaning 1, 3, 5, 7... and $u=\frac{1}{2}-\dfrac{x}{\pi}$.

28. If we need $\Lambda(qx)$ in factors, in the same case of c too large, we take $\Theta(ry)$ to gain first $\Lambda^0(q,x)$ and afterwards change x to $\frac{1}{2}\pi-x$ on both sides of the equation, or rather u into $\frac{1}{2}-u$.

I believe the result to be

$$\phi(r).\Lambda(qx)=\sqrt{a}.r^{u^2}.\mathfrak{P}(1-r^{n\pm u});\ \text{with } n=1,3,5\ldots.$$

Observe the negative sign before r, under the symbol $\mathfrak{P}$.

29. Article 27 now avails with $C.G(c\omega)=\dfrac{d.\Theta(q,x)}{\Theta(qxdx)}$,

or simpler $\qquad =\dfrac{d}{dx}.\log\Theta(q,x).$

But it may be easier to dismiss all *constant factors*, and simply to write $\Theta(qx)$ *proportional to* $r^{u^2}.\mathfrak{P}(1+r^{n\pm 2u})\ldots$ since $r=\epsilon^{-a\pi}$, with t any whatever,

$$(1+r^{2t})=r^t(r^{-t}+r^t)=r^t(\epsilon^{+a\pi t}+\epsilon^{-a\pi t}),$$

proportional to Cos $a\pi t$.

Put $2t=n\pm 2u,\ t=\frac{1}{2}n\pm u$, then $\Theta(qx)$ *varies* as

$$r^{u^2}.\mathfrak{P}.\mathrm{Cos}.(a\pi.\overline{\tfrac{1}{2}n\pm u}),$$

or with N an unknown constant, probably a function of q,

$$\Theta(q,x)=N.r^{u^2}.\mathfrak{P}.\mathrm{Cos}(a\pi.\overline{\tfrac{1}{2}n\pm u}).$$

Take *log* and differentiate, then N vanishes, and

$$d.\log\Theta(q,x)=d.u^2.\log r+\Sigma d\log\mathrm{Cos}(a\pi.\overline{\tfrac{1}{2}n\pm u}).$$

But whatever variable k means

$$d\log\mathrm{Cos}.k=\frac{d\,\mathrm{Cos}\,k}{\mathrm{Cos}\,k}=\mathrm{Tan}\,k.dk,$$

and if
$$k = a\pi . (\tfrac{1}{2}n \pm u),$$

$$dk = \pm\, a\pi . du = \mp\, a\pi . \frac{dx}{\pi} = \mp\, a\,dx.$$

Also
$$d . u^2 = 2u\,du = -\left(1 - \frac{2x}{\pi}\right)\frac{dx}{\pi};\ \log r = -a\pi,$$

$$\therefore\ \frac{d}{dx} . \log . \Theta(q, x) = a\left(1 - \frac{2x}{\pi}\right) + \Sigma . \operatorname{Tan} k . \frac{dk}{dx} .$$

In this last Σ we may write separately the two terms in which n is the same but u either $+$ or $-$; k is $a\pi . (\tfrac{1}{2}n + u)$ and $a\pi(\tfrac{1}{2}n - u)$, and dk is $-a\,dx$ and $+a\,dx$.

When dx is removed the sum of the two terms is

$$-a . \operatorname{Tan}(a\pi . \overline{\tfrac{1}{2}n + u}) + a \operatorname{Tan}(a\pi . \overline{\tfrac{1}{2}n - u}).$$

We may add *zero* $= a - a$ to the pair; then *with* ח *for* $1 - $ *Tan*, we find
$$a\left\{\text{ח}(a\pi . \overline{\tfrac{1}{2}n + u}) - \text{ח}(a\pi . \overline{\tfrac{1}{2}n - u})\right\},$$
for the general term of Σ, with n for 1, 3, 5, 7.

Also $\dfrac{d}{dx} . \log\Theta(q, x) = C . G(c\omega)$. Replacing $\tfrac{1}{2} - \dfrac{x}{\pi}$ for u, the pair of terms become

$$a\left\{\text{ח}\left(a . \overline{\frac{n+1}{2}\pi - x}\right) - \text{ח}\left(a . \overline{\frac{n-1}{2}\pi + x}\right)\right\}.$$

Divide both sides of our equation by $a = \dfrac{C}{B}$; then, in full

$$\begin{aligned} B . G(c\omega) = 1 - \frac{2x}{\pi} &- \text{ח}(ax) \\ &+ \text{ח}(a . \overline{\pi - x}) - \text{ח}(a . \overline{\pi + x}) \\ &+ \text{ח}(a . \overline{2\pi - x}) - \text{ח}(a . \overline{2\pi + x}) \\ &+ \text{ח}(a . \overline{3\pi - x}) - \text{ח}(a . \overline{3\pi + x}) + \&c. \end{aligned}$$

If we transpose $-\dfrac{2x}{\pi}$, the left hand becomes $B . J(c\omega)$.

This seems to complete the case of c too large. Another form for $\daleth(c\omega)$, *analogous to this last article*, is possible, but seems to have no advantage over Art. 25.

The last formula rightly shows $G(c\omega)$ to vanish when $x = \tfrac{1}{2}\pi$ or $x = \pi$. But in second quadrant G is negative, and the x ought never to exceed $\tfrac{1}{2}\pi$.

CHAPTER IX.

THE HIGHER SCALES.

1. OUR two last Chapters have anticipated that which to *computers* was the chief interest of these scales; but these are needed to complete the Science. We had learned from Lagrange the scale of Index 2, of Legendre the scale of Index 3; Jacobi generalizes and shows the scale of Index n. Legendre joyfully comments, that the more general argument is the clearest and easiest. Here we take advantage of our past equations to deduce the wider theory of the scales.

In Ch. VII. Art. 8, divide $\Lambda(q^n, nx)$ by $\Theta(q^n, nx)$, then

$$\frac{\Lambda(q^n, nx)}{\Theta(q^n, nx)} = \frac{\Lambda}{\Theta} \cdot \frac{\Lambda_1}{\Theta_1} \cdot \frac{\Lambda_2}{\Theta_2} \cdots \frac{\Lambda_{n-1}}{\Theta_{n-1}};$$

in which $\dfrac{\Lambda}{\Theta} = \sqrt{c}\sin\omega$, and we assume appropriate arcs $\omega', \omega'', \ldots \omega^{(n-1)}$ for the other factors, corresponding with

$$x + \frac{\pi}{n}, \quad x + \frac{2\pi}{n}, \ldots x + \frac{(n-1)\pi}{n};$$

and n may be even or odd.

We may assume h with θ related to nx and q^n, as c with ω to x and q.

$\therefore$ from $$\sqrt{c}\sin\omega = \frac{\Lambda(q, x)}{\Theta(q, x)},$$

we deduce $$\sqrt{h}\sin\theta = \frac{\Lambda(q^n, nx)}{\Theta(q^n, nx)};$$

and $\sqrt{h}\sin\theta = \sqrt{c}\sin\omega \,.\, \sqrt{c}\sin\omega' \,.\, \sqrt{c}\sin\omega'' \ldots \sqrt{c}\sin\omega^{(n-1)}$; (A).

As C means $\dfrac{F_c}{\frac{1}{2}\pi}$, naturally H means $\dfrac{F_h}{\frac{1}{2}\pi}$; and as $F(c\omega) = Cx$, so

$$F(h\theta) = H.(nx),$$

or $$\frac{F(h,\theta)}{H} = nx = n.\frac{F(c\omega)}{C};$$

and if we write $$F(c\omega) = \mu.F(h,\theta)$$

in conformity to the earlier scales, we have $C = n\mu H$.

Our first business is, to express the right hand of (A) in functions of $\sin\omega$.

2. When n is odd $= 2m+1$, we may separate the factor $\sin\omega$, and join the $2m$ factors that remain into pairs by introversion. Writing

$$\sin\omega = \sin|_c(x);$$

we have

$$\sqrt{h}\sin\theta = \sqrt{c^n}.\sin|_c(x).\sin|_c\left(x+\frac{\pi}{n}\right).\sin|_c\left(x+\frac{(n-1)\pi}{n}\right)$$

$$\sin|_c\left(x+\frac{2\pi}{n}\right).\sin|_c\left(x+\frac{(n-2)\pi}{n}\right)$$

$$\sin|_c\left(x+\frac{3\pi}{n}\right).\sin|_c\left(x+\frac{(n-3)\pi}{n}\right)\ldots\&c.$$

Now $$\sin|_c\left(x+\frac{(n-r)\pi}{n}\right) = \sin|_c\left(\frac{r\pi}{n}-x\right),$$

and by IV. 8 above

$$\sin|_c(y+x)\sin|_c(y-x) = \frac{\sin^2|_c(y) - \sin^2|_c(x)}{1-c^2\sin^2|_c(y).\sin^2|_c(x)};$$

thus each pair of factors has the form

$$\frac{\sin^2\alpha_{2p} - \sin^2\omega}{1-c^2\sin^2\alpha_{2p}\sin^2\omega};$$

if $$F(c\alpha_p) = \frac{p}{n}F_c;$$

for $$\omega = {}_c(x),\quad \tfrac{1}{2}\pi = {}_c(\tfrac{1}{2}\pi),\quad \alpha_{2p} = {}_c\left(\frac{p}{n}\pi\right).$$

Thus $$\sqrt{h}.\sin\theta = \sqrt{c^n}.\sin\omega.\mathfrak{P}\frac{\sin^2\alpha_{2p}-\sin^2\omega}{1-c^2\sin^2\alpha_{2p}.\sin^2\omega},$$

where p means 1, 2, 3 ... up to $\frac{n-1}{2}$ or m an integer.

3. From the last we readily deduce $\cos\theta$. Let

$$V = \mathfrak{P}(1-c^2\sin^2\alpha_{2p}\sin^2\omega)$$

and $$U=\sqrt{\frac{c^n}{h}}\sin\omega\,.\,\mathfrak{P}\,(\sin^2\alpha_{2\nu}-\sin^2\omega),$$

$$\therefore\quad \sin\theta=\frac{U}{V},$$

in which U and V are integer fractions of $\sin\omega$; U *odd*, of degree n, V *even*, of degree $n-1$ (where $n=2m+1$).

Hence $$1\pm\sin\theta=\frac{V\pm U}{V},$$

and $1-\sin\theta$ vanishes when $\theta=n\,.\,\frac{1}{2}\pi$, and $\omega=\frac{1}{2}\pi$; so then does $V-U$; hence it has factors by taking $\omega=\frac{1}{2}\pi$, $\omega=\pi$, $\omega=\frac{3}{2}\pi$, &c.

$$(\sin\alpha_1-\sin\omega)\,(\sin\alpha_3+\sin\omega)\,(\sin\alpha_5-\sin\omega)\ \&c.\ldots$$

and similarly with $1+\sin\theta$ by merely changing $\sin\omega$ to $-\sin\omega$; so that its factors are

$$(\sin\alpha_1+\sin\omega)\,(\sin\alpha_3-\sin\omega)\,(\sin\alpha_3+\sin\omega)\ldots\&c.$$

Multiply these together, write M^2 as an unknown constant factor,

$$\therefore\quad V^2\cos^2\theta=M^2\,.\,(\sin^2\alpha_1-\sin^2\omega)\,(\sin^2\alpha_3-\sin^2\omega)$$
$$(\sin^2\alpha_5-\sin^2\alpha)\ldots(\sin^2\alpha_{2n-1}-\sin^2\omega);$$

since n is odd, *one* factor here is $\sin^2\alpha_n-\sin^2\omega$, that is $1-\sin^2\omega$ or $\cos^2\omega$. Also on each side of this factor

$$\sin\alpha_{n-2}=\sin\alpha_{n+2};\quad \sin\alpha_{n-4}=\sin\alpha_{n+4},\ \ldots;\ \ldots\ldots$$

since $$\sin\alpha_{n\pm 2r}=\cos\left(\frac{r}{n}\,.\,\frac{\pi}{2}\right).$$

Thus every factor except $1-\sin^2\omega$ is repeated, and we find

$$V\cos\theta=M\,.\cos\omega\,(\sin^2\alpha_1-\sin^2\omega)\,(\sin\alpha_3-\sin^2\omega)\ldots(\sin\alpha_{n-2}-\sin^2\omega).$$

To find M, make $\omega=0$, $\cos\omega=1$; $V=1$, $\theta=0$, $\cos\theta=1$,

$$\therefore\quad 1=M\,.\sin^2\alpha_1\,.\sin^2\alpha_3\ldots\sin^2\alpha_{n-2}.$$

[*To catch the eye*, here and elsewhere I write β rather than α when the index is *even*.] We may also find $1+\sin\theta\div 1-\sin\theta$.

Eliminate M,

$$\therefore\quad V\cos\theta=\cos\omega\left(1-\frac{\sin^2\omega}{\sin^2\alpha_1}\right)\left(1-\frac{\sin^2\omega}{\sin\alpha_3}\right)\ldots\left(1-\frac{\sin^2\omega}{\sin^2\alpha_{n-2}}\right)\quad(1);$$

$$V\sin\theta=\frac{\sin\omega}{N}\,.\,\left(1-\frac{\sin^2\omega}{\sin^2\beta_2}\right)\left(1-\frac{\sin^2\omega}{\sin^2\beta_4}\right)\ldots\left(1-\frac{\sin^2\omega}{\beta_{n-1}}\right)\quad(2).$$

Make ω and θ infinitesimal;

$$\therefore\quad V=1,\quad N=\frac{\sin\omega}{\sin\theta}=\mu.$$

4. We proceed to find the formula for $\Delta(h, \theta)$, which is best obtained by the law of Double Inversion. Put $v = \sin\omega$, $y = \sin\theta$. In the discussion of Imaginary Amplitude we found that to change v into $(cv)^{-1}$ leaves $\dfrac{dv}{\sqrt{(1-v^2)}\sqrt{(1-c^2v^2)}}$ unchanged. If then in

$$dF(c\omega) = \mu dF(h\theta)$$

we *simultaneously* change

$$\sin\omega \text{ into } (c\sin\omega)^{-1} \text{ and } \sin\theta \text{ into } (h\sin\theta)^{-1},$$

it does not change the differential equation. Now in the trigonometrical relation of ω to θ the variables $\sin\omega$ and $\sin\theta$—(say v and y) are not only *zero* together, but also *infinite* together.

By this process of *double inversion*, $(cv)^{-1}$ for v, and $(hy)^{-1}$ for y, we change

$$\sqrt{\frac{1-y^2}{1-v^2}} = \mathfrak{P}\,.\,\frac{1-v^2\operatorname{cosec}^2\alpha_{2p-1}}{1-c^2v^2\sin^2\beta_{2p}}$$

first into
$$\frac{cv}{hy}\sqrt{\frac{1-h^2y^2}{1-c^2v^2}} = \mathfrak{P}\,\frac{1-c^2v^2\sin^2\alpha_{2p-1}}{c^2\sin^2\alpha_{2p-1}(\sin^2\beta_{2p}-v^2)}\,.$$

Eliminate $\dfrac{y}{v}$ by its equivalent in factors of v. Then

$$\frac{c}{h}\sqrt{\frac{1-h^2y^2}{1-c^2v^2}} = \sqrt{\frac{c^n}{h}}\,\mathfrak{P}\,.\,\frac{1-c^2v^2\sin^2\alpha_{2p-1}}{c^2\sin^2\alpha_{2p-1}(1-c^2v^2\sin^2\beta_{2p})}\,.$$

Now $\mathfrak{P}\,.\,c^2$ means c^{n-1}, since $\dfrac{n-1}{2}$ is the number of factors in $\mathfrak{P}$.

Make $v=0$, $y=0$. Then

$$\frac{c}{h} = \sqrt{\frac{c^n}{h}}\,.\,\frac{1}{c^{n-1}\mathfrak{P}\sin^2\alpha_{2p-1}}$$

or
$$\sqrt{h} = \sqrt{c^n}\,.\,\sin^2\alpha_1\,.\,\sin^2\alpha_3\,\ldots\,\sin^2\alpha_{n-2};$$

which with
$$\sqrt{h} = \sqrt{c^n}\,.\,\mu\mathfrak{P}\sin^2\beta$$

gives
$$\sqrt{\mu} = \mathfrak{P}\,.\,\frac{\sin\alpha_{2p-1}}{\sin\beta_{2p}}\,.$$

Also we have finally,

$$\frac{\Delta(h\theta)}{\Delta(c\omega)} = \mathfrak{P}\,.\,\frac{1-c^2\sin^2\omega\sin^2\alpha_{2p-1}}{1-c^2\sin^2\omega\sin^2\beta_{2p}} \qquad (3).$$

The equations marked (1), (2), (3) are the main results of Jacobi's First Theorem.

5. The last equation leads to a new integration of

$$\frac{\mu d\theta}{d\omega} = \frac{\Delta(h,\ \theta)}{\Delta(c,\ \omega)},$$

by giving to this surd its value just obtained. Algebraic analysis shows that with unknown constants M_{2p} and N

$$\mathfrak{P}\,\frac{1 - c^2v^2\sin^2\alpha}{1 - c^2v^2\sin^2\beta}$$

is resolvible into $$N + \Sigma\frac{M_{2p}}{1 - c^2v^2\sin^2\beta_{2p}};$$

and we may find M_{2p} by an easier process than that of Legendre. But first, make v infinite, $\therefore\ N = \mathfrak{P}\,\dfrac{\sin^2\alpha}{\sin^2\beta}$.

We already know that this means $N = \mu$. Next, write k for $c^2\sin^2\beta_{2p}$. Multiply by $1 - k^2v^2$, and *after* the multiplication make

$$1 - k^2v^2 = 0,$$

$\therefore\ M_{2p}$ alone remains on the right hand. On the left you have the

equivalent $$(1 - k^2v^2)\,.\,\mu\frac{d\theta}{d\omega},$$

or $$\frac{\mu\,.\,(1 - k^2v^2)}{\left(\dfrac{d\omega}{d\theta}\right)},$$

to be taken when $$v^2 = \frac{1}{k^2},$$

$\therefore$ when $y = \infty$ and $$\frac{d\omega}{d\theta} = 0,$$

which gives $$M_{2p} = \frac{0}{0}.$$

Differentiate numerator and denominator, which gives

$$\frac{-\,2\mu\,.\,k^2\,.\,vdv}{d\,.\left(\dfrac{d\omega}{d\theta}\right)}.$$

But $vdv = \sin\omega\cos\omega d\omega$, so that

$$M_{2p} = \frac{-\,2\mu\,.\,k^2\sin\omega\cos\omega}{\dfrac{1}{d\theta}\,d\,.\left(\dfrac{d\omega}{d\theta}\right)},$$

or
$$\frac{-2\mu . k^2 . \sin\omega\cos\omega}{\frac{d\theta}{d\omega} . \frac{d^2\omega}{d\theta^2}},$$

in which we are to make $v^2 = k^{-2}$. The numerator is

$$-2\mu k^2 . v\sqrt{(1-v^2)},$$

which here becomes $\quad -2\mu k^2 . k^{-1}\sqrt{(1-k^{-2})},$

or $\quad -2\mu\sqrt{-1}\sqrt{(1-k^2)},$

that is, $\quad -\sqrt{-1} . 2\mu . \Delta(c\beta_{2p}).$

The denominator $= \frac{d}{d\theta} . \log\left(\frac{d\omega}{d\theta}\right) = \frac{d}{d\theta} . \log . \left\{\mu . \frac{\Delta(c\omega)}{\Delta(h\theta)}\right\}.$

Since at this crisis $\quad \frac{d\omega}{d\theta} = 0,$

we may treat ω as constant in this differentiation which leaves only

$$-\frac{d}{d\theta} . \log\Delta(h, \theta),$$

or
$$\frac{h^2\sin\theta\cos\theta}{1-h^2\sin^2\theta},$$

or
$$\frac{h^2y\sqrt{(1-y^2)}}{1-h^2y^2},$$

which, when $y=\infty$, converges to $\frac{\sqrt{(1-y^2)}}{-y}$ or $-\sqrt{-1}$. Finally

$$M_{2p} = 2\mu . \Delta(c\beta_{2p}),$$

whence, dividing by μ,

$$\frac{d\theta}{d\omega} = 1 + \Sigma\frac{2\Delta(c\beta_{2p})}{1 - c^2\sin^2\omega\sin^2\beta_{2p}}.$$

This is directly integrable,

$$\tfrac{1}{2}(\theta-\omega) = \Sigma\tan^{-1} . \{\Delta(c\beta_{2p}) . \tan\omega\},$$

giving an elegant generalization of the case, where $n = 3$, and

$$\tan . \tfrac{1}{2}(\theta-\omega) = \Delta(c\beta)\tan\omega.$$

6. We pass from the 1st to the 2nd Theorem of Jacobi by making $\sin\omega = \sqrt{-1}\tan\psi$, and $\sin\theta = \sqrt{-1}\tan\chi$. It involves only trigonometrical routine. Nothing can be added to the lucid treatment of Legendre's "Supplements." The main interest here is, that ψ and χ reach $\tfrac{1}{2}\pi$ together, so that, with $F_c = n\mu . F_h$ from the former theorem,

we have $F_b = \mu F_g$ from the latter; if then ρ be the promodulus for c, and ρ_1 for h, we have now $\rho_1 = n\rho$. This completes the analogy.

The relations of the constants are exhaustively developed in Legendre.

7. If ω and θ both vary together in x, then by Ch. III. Art. 9,

$$\delta x = \frac{dx}{d\omega}\,\delta\omega + \frac{dx}{d\rho}\,.\,\delta\rho = \frac{\delta\omega}{C\Delta\,(c\omega)} + C\,.\,G\,(c\omega^0)\,.\,\delta\rho.$$

So in the scale with index n, where $n\rho$, nx replace ρ, x,

$$\delta\,(nx) = \frac{\delta\theta}{H\Delta\,(h\theta)} + H\,.\,G\,(h\theta^0)\,.\,\delta\,(n\rho).$$

Identify x and ρ in the two equations and *make ω constant*, $\delta\omega = 0$.

Let L^0 stand for $CG\,(c\omega^0) - HG\,(h\theta^0)$, we find

$$L^0 = [nH\,.\,\Delta\,(h\theta)]^{-1}\,.\,\frac{d\theta}{d\rho}\,.$$

Our first business is to deduce the value of $\frac{d\theta}{d\rho}$ (when ω is constant) from the integral which closes Art. 5. Differentiating it for c and ρ variable,
$$\frac{d\theta}{d\rho} = 2\Sigma\,\frac{\tan\omega}{1 + \Delta^2\,(c\beta)\tan^2\omega}\,.\,\frac{d\Delta\,(c,\beta)}{d\rho}\,;$$
where β means $\alpha_2, \alpha_4, \alpha_6 \dots \alpha_{n-1}$. By the general law of functions of two elements, our $\frac{d\omega}{d\rho}$ (with x const.)

$$= -\frac{dx}{d\rho}\ (\text{when } \omega \text{ is } const.) \times \frac{d\omega}{dx}\ (\text{with } \rho \text{ const.}),$$

or
$$\frac{d\omega}{d\rho} = -C\,.\,G\,(c\omega^0)\,.\,C\Delta\,(c\omega).$$

Make $\omega = \beta_{2p}$, $\quad \therefore \quad \frac{d\beta_{2p}}{d\rho} = -C^2\,.\,G(c\beta_{n-2p})\,.\,\Delta\,(c\beta_{2p}).$

Also by Ch. II. Art. 8, $\qquad \frac{-dc}{d\rho} = b^2cC^2,$

$$\therefore\quad \frac{d\Delta\,(c\beta)}{d\rho} = \frac{-c\sin\beta}{\Delta\,(c\beta)}\,.\,\frac{d\,.\,c\sin\beta}{d\rho}$$
$$= \frac{-c\sin\beta}{\Delta\,(c\beta)}\left\{\sin\beta\,\frac{dc}{d\rho} + c\cos\beta\,.\,\frac{d\beta}{d\rho}\right\},$$

which further $= + \dfrac{c \sin \beta_{2p}}{\Delta(c\beta_{2p})} \{b^2 c C^2 \sin \beta_{2p} + c \cos \beta_{2p} . C^2 G(c\beta_{n-2p}) \Delta(c\beta_{2p})\}$.
Divide this by $C^2 c^2$ and *call the quotient* A_{2p} for conciseness. It is presumed to vary with $2p$, the index of β. Also

$$\frac{\tan \omega}{1 + \Delta^2(c\beta)\tan^2 \omega} = \frac{\sin \omega \cos \omega}{1 - c^2 \sin^2 \beta \sin^2 \omega}.$$

Then $$\frac{d\theta}{d\rho} = 2C^2 c^2 \sin \omega \cos \omega \, \Sigma \frac{A_{2p}}{1 - c^2 \sin^2 \beta_{2p} \sin^2 \omega}:$$
where A_{2p} concerns us little.

For we reduce the fractions under Σ to a common denominator. The numerator is *an integer function* of $\sin^2 \omega$, of the degree $\sin^{n-3} \omega$. Call it $\phi(\sin^2 \omega)$. To obtain L^0, divide $\dfrac{d\theta}{d\rho}$ by $nH . \Delta(h, \theta)$ and give to $\Delta(h, \theta)$ on the right from the close of Art. 4 its equivalent

$$\Delta(c\omega)\, \mathfrak{P} . \frac{1 - c^2 \sin^2 \omega \sin^2 \alpha_{2p-1}}{1 - c^2 \sin^2 \omega \sin^2 \beta_{2p}}.$$

Observe that $C^2 = C . n\mu H$; then

$$L^0 = \frac{2Cc^2 \sin \omega \cos \omega}{\Delta(c\omega)} . \frac{\mu\phi(\sin^2 \omega)}{\mathfrak{P}(1 - c^2 \sin^2 \omega \sin^2 \alpha_{2p-1})},$$

and the last fraction is again resolvible into the form

$$\Sigma . \frac{P_{2p-1}}{1 - c^2 \sin^2 \omega \sin^2 \alpha_{2p-1}};$$

where P_{2p-1} does not contain ω. Further

$$\frac{\cos \omega}{\Delta(c\omega)} = \sin \omega^0,$$

hence more simply $$L^0 = C . \Sigma \frac{2c^2 \sin \omega \sin \omega^0 . P_{2p-1}}{1 - c^2 \sin^2 \omega \sin^2 \alpha_{2p-1}}.$$

[Afterwards, by retracing our steps, we can prove that P_{2p-1} here $= \dfrac{1}{n} \cos^2 \alpha_{2p-1}$.]

8. Our next step is to change ω^0 into ω. Observe that this changes Mesonome x into $\frac{1}{2}\pi - x$, and nx Mesonome of θ into

$$n(\tfrac{1}{2}\pi - x),$$

or into $$\frac{n-1}{2} . \pi + (\tfrac{1}{2}\pi - x),$$

and here $\frac{n-1}{2}$ is integer. But $G(h, \theta)$ remains unchanged if $m\pi$ be added or taken from its amplitude or mesonome; therefore to change here ω^0 to ω, changes θ^0 to θ, and L will represent the new value of L^0.

[To examine more closely,

$$F(c\omega^0) = F_c - F(c\omega) = n\mu F_h - \mu F(h\theta) = (n-1)\,\mu F_h + \mu F(h\theta^0)$$
$$= \mu\left\{F\left(h, \frac{n-1}{2}\pi\right) + F(h, \theta^0)\right\} = \mu F\left(h, \theta^0 + \frac{n-1}{2}\pi\right).$$

Thus to change ω^0 into ω, changes θ^0 into $\theta + \frac{n-1}{2}\pi$, but the addition of $\frac{n-1}{2}\pi$ to the amplitude in no respect affects the Nomiscus. The double change of ω^0 to ω and θ^0 to θ is thus justified, when n is odd.]

Put now

$$L = CG(c\omega) - HG(h\theta) = C\,.\,\Sigma\,\frac{2c^2 \sin\omega^0 \sin\omega\,.\,P}{1 - c^2\sin^2\alpha\,.\sin^2\omega^0}.$$

In the denominator for $\sin^2\omega^0$ write its equal $\frac{\cos^2\omega}{\Delta^2(c\omega)}$ and the denominator becomes

$$\frac{\Delta^2(c\omega) - c^2\sin^2\alpha.\cos^2\omega}{\Delta^2(c\omega)},$$

or

$$\frac{\Delta^2(c\alpha) - c^2\cos^2\alpha\sin^2\omega}{\Delta^2(c\omega)} = \frac{\Delta^2(c\alpha)}{\Delta^2(c\omega)}\{1 - c^2\sin^2\beta\sin^2\omega\},$$

since $\frac{\cos^2\alpha}{\Delta^2(c\alpha)} = \sin^2\beta$. Finally if, with a new constant M,

$$P = M,\ \sin^2\beta,$$
$$L = C\,.\,\Sigma\,.\,M\,.\,\frac{2c^2\sin^2\beta\sin\omega\cos\omega\Delta(c\omega)}{1 - c^2\sin^2\beta\sin^2\omega};$$

in which M, *though free from* ω, may have various interpretations for each separate value of β, to which Σ alludes.

[In Legendre's Scale, Ch. VI. Art. 21 (p. 43 above), where this formula has one term only, we have the numerical fraction $\frac{1}{3}$ corresponding to the M of this article.]

Multiply the last equation by

$$\frac{dF(c\omega)}{C} = \frac{dF(h\theta)}{nH} = \frac{d\omega}{\Delta(c\omega)}$$

and integrate, then

$$ℸ(c\omega) - \frac{1}{n}ℸ(h\theta) = -\Sigma\,.\,M\,.\log(1 - c^2\sin^2\beta\sin^2\omega),$$

a striking result, where nothing remains unknown but the set of constants $M_2 M_4 \dots M_{n-1}$.

Research of the constants M.

9. The research of these constants may seem too elaborate, yet on the way it reveals a few other things. Let $f(\omega\beta)$ mean

$$\frac{2c^2 \sin^2\beta \sin\omega \cos\omega \Delta(c\omega)}{1 - c^2 \sin^2\beta \sin^2\omega}$$

then we have $L = \Sigma . C . Mf(\omega, \beta)$. Write

$$\left.\begin{aligned} F(c\eta) &= F(c\omega) + F(c, \beta) \\ F(c\psi) &= F(c\omega) - F(c\beta) \end{aligned}\right\},$$

then by Euler's Integrals

$$\sin\eta - \sin\psi = \frac{2\sin^2\beta \cos\Delta(c\omega)}{1 - c^2\sin^2\beta \sin^2\omega}.$$

$$\therefore f(\omega\beta) = c^2 \sin\beta \sin\omega (\sin\eta - \sin\psi).$$

Again $\quad E\omega + E\beta - E\eta = c^2 \sin\beta \sin\omega \sin\eta$;

$$E\omega - E\beta - E\psi = -c^2 \sin\beta \sin\omega \sin\psi.$$

We infer, $\quad f(\omega\beta) = 2E\omega - E\eta - E\psi.$

Differentiate with c and β constant, also ω principal variable

$$f'(\omega\beta) = 2\Delta\omega - \Delta\eta . \frac{d\eta}{d\omega} - \Delta\psi . \frac{d\psi}{d\omega}.$$

But
$$\frac{d\eta}{\Delta\eta} = \frac{d\omega}{\Delta\omega} = \frac{d\psi}{\Delta\psi},$$

$$\therefore \Delta\omega . f'(\omega\beta) = 2\Delta^2\omega - \Delta^2\eta - \Delta^2\psi = c^2(\sin^2\eta + \sin^2\psi - 2\sin^2\omega).$$

On the other hand

$$\frac{dL}{d\omega} = C\frac{dG(c\omega)}{d\omega} - H . \frac{dG(h\theta)}{d\omega}$$

$$= C\left\{\Delta(c\omega) - \frac{\aleph_c}{\Delta(c\omega)}\right\} - H\left\{\Delta(h, \theta) - \frac{\aleph_h}{\Delta(h, \theta)}\right\}\frac{d\theta}{d\omega},$$

$$\therefore \Delta(c\omega) . \frac{dL}{d\omega} = C\{\Delta^2(c\omega) - \aleph_c\} - H\{\Delta^2(h, \theta) - \aleph_h\}\mu^{-1},$$

since
$$\frac{d\theta}{d\omega} = \mu^{-1} . \frac{\Delta(h, \theta)}{\Delta(c\omega)}.$$

Thus giving to $\frac{dL}{d\omega}$ and $f'(\omega\beta)$ their values and dividing by $C = n\mu H$

$$(\Delta^2 - \aleph_c) - \frac{1}{n\mu^2} \cdot (\Delta_1^2 - \aleph_h) = c^2 \Sigma . M (\sin^2\eta + \sin^2\psi - 2\sin^2\omega).$$

10. Assume
$$\left.\begin{aligned} s &= (\aleph_c - b^2) - \frac{1}{n\mu^2}(\aleph_h - g^2) \\ t &= (1 - \aleph_c) - \frac{1}{n\mu^2}(1 - \aleph_h) \end{aligned}\right\} .$$

Then $s + t = c^2 - \frac{h^2}{n\mu^2}$.

Now $\Delta^2 - \aleph_c = (1 - \aleph_c) - c^2\sin^2\omega$, $\Delta_1^2 - \aleph_h = (1 - \aleph_h) - h^2\sin^2\theta$,

$$\therefore t - c^2\sin^2\omega + \frac{h^2\sin^2\theta}{n\mu^2}$$

$$\left[\text{or } c^2\cos^2\omega - \frac{h^2\cos^2\theta}{n\mu^2} - s\right] = c^2\Sigma . M (\sin^2\eta + \sin^2\psi - 2\sin^2\omega).$$

Since either ω or θ is arbitrary, let $\theta = p . \frac{1}{2}\pi$, $\therefore \omega = \alpha_p$, and for clearness we must write for the values of M the series $M_2 M_4 \ldots M_{n-1}$. Also we must write α, the index being doubtful. Then $\eta = \alpha_{p+2m}$, $\psi = \alpha_{p-2m}$ leaving m undecided. When p is odd, $\cos\theta = 0$, $\sin^2\theta = 1$. When p is even, $\cos^2\theta = 1$, $\sin\theta = 0$.

Thus, with *p odd*,

$$c^2\cos^2\alpha_p - s = c^2\Sigma \{M(\sin^2\alpha_{p+2m} + \sin^2\alpha_{p-2m} - 2\sin^2\alpha_p)\}.$$

But with *p even*, the left hand is expressed by $t - c^2\sin^2\alpha_p$. In this way we get rid of h entirely. By making $p = 1, 2, 3 \ldots (n-1)$ we have $(n-1)$ [or say, $2r$] different equations, and the number of unknown constants, counting s and t, is $r + 2$. *When $n = 3$, we have only 2 equations and 3 quantities unknown; so that our equations do not suffice.* When $n = 5$, $2r = 4 = r + 2$, which shows 4 equations with 4. [To fix ideas, solve for $n = 5$. Make $p = 1$, 3 and 2, 4. Write $a'b'c'd'$ for $\sin^2\alpha_1 \sin^2\alpha_2 \sin^2\alpha_3 \sin^2\alpha_4$. Then

$$\left.\begin{aligned} c^2(1 - a') - s &= c^2M_2(c' - a') + c^2M_4(1 + c' - 2a') \\ c^2(1 - c') - s &= c^2M_2(1 + a' - 2c') + c^2M_4(a' - c') \\ t - c^2b' &= c^2M_2(a' - 2b') + c^2M_4 . (a' - b') \\ t - c^2d' &= c^2M_2(b' - d') + c^2M_4(b' - 2d') \end{aligned}\right\}$$

Eliminate s from the two first and t from the two last, and you can divide by c^2. It remains to find M_2 and M_4 from the two resulting equations, *viz.* from

$$(c'-a') = M_2(3c'-2a'-1) + M_4(1+2c'-3a'),$$
$$(d'-b') = M_2(2d'-3b') + M_4(3d'-2b').$$

In each equation the sum of the two coefficients on the right is just 5 times the quantity on the left. This puts in evidence that $M_2=\frac{1}{5}$ and $M_4=\frac{1}{5}$ is one solution; and therefore here the *only* solution.]

Since also in Legendre's scale we can prove the M to be $\frac{1}{3}$, we are prepared to expect that in the scale of Index n every M of this series will be exactly $=\frac{1}{n}$.

11. We have $(n-1)$ equations of which the odd ones contain s and the even ones t. Subtract every $(p+2)^{\text{th}}$ equation from every p^{th}, and $(n-3)$ equations will remain, free from s and t. The left hand, when p is odd, will become

$$c^2(\cos^2\alpha_p - \cos^2\alpha_{p+2}) \text{ or } c^2(\sin^2\alpha_{p+2} - \sin^2\alpha_p);$$

which is exactly what we have on the left when p is even. No distinction between the two cases remain. Let us denote

$$\sin^2\alpha_{p+2} - \sin^2\alpha_p$$

by ϟ_p. Divide both sides by c^2. Then every equation of the series is represented by

$$\text{ϟ}_p = \Sigma\,.\,M_{2m}(2\text{ϟ}_p - \text{ϟ}_{p+2m} - \text{ϟ}_{p-2m}),$$

in *each* of which m has the values $1, 2, 3 \ldots r$, where $2r+1=n$.

If in r (or more) *linear* equations with r unknown quantities we find a system of values for these which fulfils all the equations, we have certainly alighted on the true values; for linear equtaions admit no second solution. And *I say*, here, every $M=\frac{1}{n}$. *In proof*, let us introduce this value, and test the result, leaving p arbitrary. Making $M_{2m}=\frac{1}{n}$, we pass it outside of Σ,

then
$$n\text{ϟ}_p = \Sigma(2\text{ϟ}_p) - \Sigma(\text{ϟ}_{p+2m} + \text{ϟ}_{p-2m}).$$

But $\Sigma\,.\,2\text{ϟ}_p$ for r terms gives $2r\,.\,\text{ϟ}_p$ and $n-2r=1$, which reduces the equation to

$$\text{ϟ}_p = -\Sigma(\text{ϟ}_{p+2m} + \text{ϟ}_{p-2m}),$$

which we must test. Now $\Sigma\text{ל}_{p+2m}$ means

$$\text{ל}_{p+2}+\text{ל}_{p+4}+\text{ל}_{p+6}+\ldots+\text{ל}_{p+r}.$$

Look to the definition of ל and you find the sum of these r terms to be

$$\left.\begin{array}{l}\sin^2\alpha_{p+2r+2}-\sin^2\alpha_{p+2},\\ \text{Again, } \Sigma\text{ל}_{p-2m} \text{ means}\\ \text{ל}_{p-2}+\text{ל}_{p-4}+\ldots+\text{ל}_{p-2r} \text{ or } \sin^2\alpha_p-\sin^2\alpha_{p-2r}\end{array}\right\}.$$

But from the nature of α, we get $\alpha_{-p}=-\alpha_p$, so that

$$\sin^2\alpha_{p-2r}=\sin^2\alpha_{2r-p},$$

$$\therefore\ \Sigma\,(\text{ל}_{p+2m}+\text{ל}_{p-2m})=-(\sin^2\alpha_{p+2}-\sin^2\alpha_p)+(\sin^2\alpha_{p+2r+2}-\sin^2\alpha_{2r-p}).$$

The two last indices are supplementary; for

$$(p+2r+2)+(2r-p)=(p+n+1)+(n-1-p)=2n;$$

therefore the difference of the terms is zero. Also

$$\sin^2\alpha_{p+2}-\sin^2\alpha_p=\text{ל}_p;$$

thus the equation remaining is simply $\Sigma\,(\text{ל}_{p+2m}+\text{ל}_{p-2m})=-\text{ל}_p$, which is exactly what alone we had now to prove. The equation being thus verified with p arbitrary, it is established universally, and *we ascertain that every* $M=\frac{1}{n}$.

Finally then

$$\daleth(c\omega)-\frac{1}{n}\daleth(h\theta)=-\frac{1}{n}\Sigma\log(1-c^2\sin^2\beta_{2m}\sin^2\omega),$$

where
$$m=1,\,2,\,3\ldots\frac{n-1}{2}.$$

12. We now learn also the values of s and t. Put

$$S=\frac{1}{n}\Sigma\,(\sin^2\alpha_{p+2m}+\sin^2\alpha_{p-2m}-2\sin^2\alpha_p),$$

then, with p *odd*, $s=c^2(\cos^2\alpha_p-S)$, and with p *even*, $t=c^2(\sin^2\alpha_p+S)$.

The values of s and t do not vary with p. It suffices to assume $p=1$ and $p=2$. Also $\Sigma\,2\sin^2\alpha_p$ means $(n-1)\sin^2\alpha_p$; also

$$\sin^2\alpha_{n-p}=\sin^2\alpha_{n+p},$$

because the indices are supplementary. Again

$$\sin^2\alpha'+\sin^2\beta'-2\sin^2\gamma'=2\cos^2\gamma'-\cos^2\alpha'-\cos^2\beta',$$

$$\therefore\ S=\frac{1}{n}\Sigma\,(\sin^2\alpha_{p+2m}+\sin^2\alpha_{p-2m})-\frac{(n-1)}{n}\sin^2\alpha_p,$$

$$\therefore \sin^2\alpha_p + S = \frac{1}{n}\{\sin^2\alpha_p + \Sigma(\sin^2\alpha_{p+2m} + \sin^2\alpha_{p-2m})\},$$

so too $$\cos^2\alpha_p - S = \frac{1}{n}\{\cos^2\alpha_p + \Sigma(\cos^2\alpha_{p+2m} + \cos^2\alpha_{p-2m})\}.$$

Put $p = 1$, $\therefore$ since $\cos^2\alpha_{1-2m} = \cos^2\alpha_{2m-1}$,

$$s = \frac{c^2}{n}\{\cos^2\alpha_1 + \Sigma(\cos^2\alpha_{2m+1} + \cos^2\alpha_{2m-1})\}.$$

Put $p = 2$, then similarly

$$t = \frac{c^2}{2n}\{\sin^2\alpha_2 + \Sigma(\sin^2\alpha_{2m+2} + \sin^2\alpha_{2m-2})\}.$$

Remember that m after Σ has the r values 1, 2, 3... $\frac{n-1}{2}$; also $\alpha_n = \frac{1}{2}\pi$, $\alpha_0 = 0$, extreme values. Then developing each Σ, you get

$$\left\{\begin{aligned} s &= \frac{2c^2}{n}(\cos^2\alpha_1 + \cos^2\alpha_3 + \dots + \cos^2\alpha_{n-2}) \\ t &= \frac{2c^2}{n}(\sin^2\alpha_2 + \sin^2\alpha_4 + \dots + \sin^2\alpha_{n-1}) \end{aligned}\right\}.$$

See the meaning of s and t in the opening of Art. 10. Also

$$s + t = c^2 - \frac{h^2}{n\mu^2}.$$

[Now too we may imitate in the scale n the procedure of VI. 15 in the scale 3. It gives $-\frac{d\log\mu}{d\rho} = C^n s$.]

13. Descent upon Legendre's Scale. The method of Art. 10 failed when $n = 3$. The process originally used by Legendre (reproduced in Ch. VI.) rests on elaborate relations of μ peculiar to that scale. It is more satisfactory to deduce them from the general theory.—Leaving M unknown in Arts. 8 and 10 above, make $n = 3$, which gives but a single M. Drop Σ, put $m = 1$, $p = 1$ with s, but $p = 2$ with t, then from

$$\left.\begin{aligned} c^2\cos^2\alpha_p - s &= c^2M(2\cos^2\alpha_p - \cos\alpha_{2m+p} - \cos^2\alpha_{2m-p}) \\ \text{and}\quad t - c^2\sin^2\alpha_p &= c^2M(\sin^2\alpha_{2m+p} + \sin^2\alpha_{2m-p} - 2\sin^2\alpha_p) \end{aligned}\right\},$$

remembering that $\alpha_0 = 0$ and $\alpha_n = \frac{1}{2}\pi$, you deduce simply

$$\left\{\begin{aligned} s &= (1 - M)\,c^2\cos^2\alpha_1 \\ t &= (1 - M)\,c^2\sin^2\alpha_2. \end{aligned}\right.$$

Next differentiate the equations for s and t in Art. 10 *before* assuming any value for θ. They are now only

$$\left.\begin{aligned} t - c^2\sin^2\omega + \frac{h^2\sin^2\theta}{3\mu^2} &= c^2M(\sin^2\eta + \sin^2\psi - 2\sin^2\omega) \\ c^2\cos^2\omega - \frac{h^2\cos^2\omega}{3\mu^2} - s &= c^2M(2\cos^2\omega - \cos^2\eta - \cos^2\psi) \end{aligned}\right\}.$$

But *one* suffices; we need only one more equation.

In differentiation, the constants s and t will vanish, but the factor M will remain. In thus removing t, let $\phi_c(\omega)$ stand for

$$\sin\omega\cos\omega\,\Delta(c\omega).$$

Differentiation gives, first,

$$-2c^2\sin\omega\cos\omega\,d\omega + \frac{h^2}{3\mu^2}\,.\,2\sin\theta\cos\theta\,d\theta$$

$$= c^2M(2\sin\eta\cos\eta\,d\eta + 2\sin\psi\cos\psi\,d\psi - 4\sin\omega\cos\omega\,d\omega).$$

But $\quad d\omega : d\eta : d\psi : d\theta = \Delta(c\omega) : \Delta(c\eta) : \Delta(c\psi) : \mu^{-1}\Delta(h\theta).$

Substitute to eliminate $d\omega$, $d\eta$, &c. Then

$$-\phi_c(\omega) + \frac{h^2}{3c^2\mu^3}\,.\,\phi_h(\theta) = M\{\phi_c(\eta) + \phi_c(\psi) - 2\phi_c(\omega)\}.$$

From this we determine M by assuming either $\omega = \alpha$ or $\omega = \beta$ [where $F(c\alpha) = \frac{1}{3}F_c$ and $F(c\beta) = \frac{2}{3}F_c$].

Thus, if $\omega = \alpha$, $\quad \theta = \frac{1}{2}\pi$, $\quad \phi_h(\theta) = 0$, $\quad \eta = \alpha_{1+2} = \frac{1}{2}\pi$, $\quad \phi_c(\eta) = 0$,

$$\psi = \alpha_{1-2} = -\alpha_1, \quad \phi_c(\psi) = -\phi_c(\alpha).$$

Hence $\quad -\phi_c(\alpha) = M\{-\phi_c(\alpha) - 2\phi_c(\alpha)\}$ or $M = \frac{1}{3}$. Q.E.D.

COR. In Legendre's scale $\begin{cases} s = \frac{2}{3}c^2\cos^2\alpha \\ t = \frac{2}{3}c^2\sin^2\beta. \end{cases}$

14. We now more easily find $G(c\alpha)$ and $G(c\beta)$. We had $L = \frac{1}{3}Cf(\omega\beta)$ which means

$$L = \tfrac{1}{3}Cc^2\sin\beta\sin\omega(\sin\eta - \sin\psi).$$

Let $\omega = \alpha$, $\theta = \frac{1}{2}\pi$, $G(h, \theta) = 0$, $\therefore$ L becomes $CG(c\alpha)$, also $\eta = \frac{1}{2}\pi$, $\psi = -\alpha$, thus

$$CG(c\alpha) = \tfrac{1}{3}Cc^2\sin\beta\sin\alpha(1 + \sin\alpha).$$

Now $\quad \sin\alpha + \cos\beta = 1, \ \cot\alpha\,.\cot\beta = b,$

$$\sin\beta = \tan\beta\cos\beta = (b^{-1}\cot\alpha)(1 - \sin\alpha),$$

$$\therefore\ \sin\beta(1 + \sin\alpha) = b^{-1}\cot\alpha\,(\cos^2\alpha),$$

and $$\sin\beta\,.\,\sin\alpha\,(1+\sin\alpha)=b^{-1}\,.\,\cos^2\alpha.$$

Hence $$G\,(c\alpha)=\tfrac{1}{3}\,.\,c^2\,.\,b^{-1}\,.\,\cos^3\alpha.$$

Again, if we make $\omega=\beta$, $\theta=\pi$, $\eta=\alpha_4$, $\sin\eta=\sin\alpha_2$ or $\sin\beta$, $\psi=0$, and you obtain $G\,(c\beta)=\tfrac{1}{3}\,c^2\sin^3\beta$.

In general

$$nG\,(c\alpha_p)=c^2\sin\alpha_p\,.\,\Sigma\,.\,\sin\alpha_{2m}\,(\sin\alpha_{p+2m}-\sin\alpha_{p-2m}),$$

where $$m=1,\ 2,\ 3\ldots\ \frac{n-1}{2}.$$

15. Indeed if instead of the amplitude we write the Mesonome after $\daleth$ and G, we can here generalize. For we had

$$Q_n\,.\,\Theta\,(q^n,\ nx)=Q\,.\,\Theta\,\Theta_1\Theta_2\ldots\Theta_{n-1};$$

also $$\daleth=\log\frac{\Theta\,(x)}{\Theta\,(0)},$$

$\therefore$ with k an unknown constant

$$\daleth_h\,(nx)=\daleth_c(x)+\daleth_c\left(x+\frac{\pi}{n}\right)+\daleth_c\left(x+\frac{2\pi}{n}\right)+\ldots+\daleth_c\left(x+\frac{n-1}{n}\pi\right)-k.$$

In the other notation $\daleth(\omega)+\daleth(\pi-\omega)=0$; in this notation then $\daleth_c(x)+\daleth_c(\pi-x)=0$. When $x=0$,

$$k=\daleth\frac{\pi}{n}+\daleth\frac{2\pi}{n}+\daleth\frac{3\pi}{n}+\ldots+\daleth\frac{n-1}{n}\pi.$$

Here the extreme terms destroy one another and when n is odd, we see $k=0$. But if n is even, there is a middle term $\daleth_c\left(\dfrac{h\pi}{n}\right)$ or $\daleth_c(\tfrac{1}{2}\pi)$, i.e. $\daleth_c$ or $-\tfrac{1}{2}\log b$.

When we differentiate, k vanishes; then

$$nH\,.\,G_h\,(nx)=C\left\{G_c\,(x)+G_c\left(x+\frac{\pi}{n}\right)+\ldots+G_c\left(x+\frac{n-1}{n}\pi\right)\right\},$$

if h is related to q^n as c to q.

This seems to supersede Legendre's elegant but laborious result when $n=5$.

He makes $$5\mu\mu'=1,\ m'^2=\frac{1}{\mu}+\frac{1}{\mu'}-2.$$

His result is

$$E_h=\frac{1}{\mu}E_h+\left(\frac{1}{2\mu'}-\frac{1}{2\mu}-m'\right)F_h.$$

16. Our work would stand thus. Making $n=5$, then observe that in s, $p=1$, and in t, $p=2$; while our m in M has the two values 1, 2 in each equation.

Also $\alpha_5 = \frac{1}{2}\pi,\ \sin\alpha_6 = \sin\alpha_4,\ \alpha_0 = 0.$

Thence
$$\left.\begin{aligned}\frac{5s}{c^2} &= 2\cos^2\alpha_1 + 2\cos^2\alpha_3\\ \frac{5t}{c^2} &= 2\sin^2\alpha_2 + 2\sin^2\alpha_4\end{aligned}\right\}.$$

From these
$$(1-\aleph_c) - \frac{1}{5\mu^2}(1-\aleph_h) = t$$
$$= \tfrac{2}{5}c^2(\sin^2\alpha_2 + \sin^2\alpha_4),$$
which with us might seem final.

17. The nearest approach to generalization for $\aleph_h$ with n any odd number, which I here can see, is by
$$\left.\begin{aligned}\frac{ns}{2c^2} &= \cos^2\alpha_1 + \cos^2\alpha_3 + \ldots\ldots + \cos^2\alpha_{n-2}\\ \frac{nt}{2c^2} &= \sin^2\alpha_2 + \sin^2\alpha_4 + \ldots\ldots + \sin^2\alpha_{n-1}\end{aligned}\right\},$$
remembering that with ρ as our *given* chief constant, $\sin^2\alpha_r$ and $\cos^2\alpha_r$ are rapidly calculable by Ch. VIII. Also $-\dfrac{d\log\mu}{d\rho} = C^2 s$, and for the left hand we replace its equivalent by
$$\Gamma(c) - n\,\Gamma(h)$$
where $\Gamma(c)$ means $C^2(\aleph_c - b^2)$: see VI. 15.

PART III.

ELLIPTIC INTEGRALS.

CHAPTER X.

On the Third Elliptic Integral.

Paranome reducible to two Elements.

1. This treatise opened abruptly, without explaining by what process we are led to the three forms called Elliptic Integrals. Perhaps the defect ought here to be repaired. We can but follow closely in the steps of Legendre.

In general if $Q = \alpha + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4$ and ϕ is a rational form, the integral $\int \phi(x, \sqrt{Q})\, dx$ is that from which we start. Evidently $\phi(x, \sqrt{Q})$ admits of being expressed as either $P + R\sqrt{Q}$ or, if we please, $P + \dfrac{R}{\sqrt{Q}}$ where P and R are rational in x. $\int P\, dx$ is known in the Elements of the Integral Calculus. Only $\int \dfrac{R}{\sqrt{Q}}\, dx$ remains to be treated.

Let
$$Q = (\zeta + 2\eta x + \theta x^2)(\lambda + 2\mu x + \nu x^2),$$
where, as we know by Algebra, $\zeta\, \eta\, \theta\, \lambda\, \mu\, \nu$ are real, if $\alpha\, \beta\, \gamma\, \delta\, \epsilon$ are all real, and the former may be found from the latter, independent of x. If then it happen that $\eta : \theta = \mu : \nu$, we may simply write $\eta = m\theta$, $\mu = m\nu$, and by assuming $y = x + m$, we at once give to Q the form $(a' + \theta y^2)(b' + \nu y^2)$ free from the term containing y. *At this we aim.* But if $\eta : \theta$ is *not* $= \mu : \nu$, assume $x = \dfrac{p + qy}{1 + y}$ and try to assume p and q constants such as will make the surd under $\sqrt{\ }$ an even function

of y. By substituting for x, we find that the two coefficients of y which we desire to vanish are

$$\zeta + \eta(p+q) + \theta(pq) = 0; \quad \lambda + \mu(p+q) + \nu(pq) = 0.$$

By hypothesis $\eta : \theta$ is *not* $= \mu : \nu$, therefore *finite* values here result for $(p+q)$ and pq. But that these values may be *real* as well as finite, it is further necessary that $(p+q)^2 - 4pq$ must be positive.

2. This condition can always be fulfilled. For if the roots of $Q=0$ are all real (say, a, b, c, e) arrange them so that $a-b$, $b-c$, $c-e$ may all be positive, when

$$Q = (x-a)(x-b)(x-c)(x-e).$$

The two equations of condition are now

$$\left\{\begin{matrix} ab - \frac{1}{2}(a+b)(p+q) + pq = 0 \\ ce - \frac{1}{2}(c+e)(p+q) + pq = 0 \end{matrix}\right\},$$

from which by mere routine you get

$$\left(\frac{p-q}{2}\right)^2 = \frac{(a-c)(a-e)(b-c)(b-e)}{(a+b-c-e)^2}$$

positive. Hence p and q are real.

But if *not all* four roots of Q be real, either $\lambda\nu - \mu^2$ or $\zeta\theta - \eta^2$ or both are positive. Suppose $\lambda\nu - \mu^2 > 0$,

$$(p-q)^2 = (p+q)^2 - 4pq = (p+q)^2 + 4\frac{\lambda + \mu(p+q)}{\nu},$$

or

$$\left(p + q + 2\frac{\mu}{\nu}\right)^2 + \frac{4}{\nu^2}(\lambda\nu - \mu^2),$$

positive,—whatever the sign of $\zeta\theta - \eta^2$. In every case therefore p and q are real. Thus our problem is reduced to $\int \frac{R'}{\sqrt{Q'}} dy$, where Q' is of the form $(a+by^2)(c+ey^2)$, and R' is rational in y.

3. Thus without loss of generality we may start anew from $\int \frac{R}{\sqrt{Q}} dx$, with Q of the form $(a+bx^2)(c+ex^2)$. Also we may put $R = M + Nx$, where M, N are *even* functions of x; and $\int \frac{Nx\,dx}{\sqrt{Q}}$ is obviously integrable by assuming $x^2 = y$. Then only $\int \frac{M}{\sqrt{Q}} dx$ remains.'

In all cases we can reduce Q to the form $Q' = (1-y^2)(1-c^2y^2)$ with $c^2 < 1$, by one or other of the following assumptions, in all of which we suppose $p^2 > q^2$.

1. If $Q = m^2(1+p^2x^2)(1+q^2x^2)$, assume $p^2x^2 = \dfrac{y^2}{1-y^2}$, $p^2 - q^2 = p^2c^2$.
2. If $Q = m^2(p^2x^2-1)(q^2x^2-1)$, assume $qx = y^{-1}$, $q = pc$.
3. If $Q = m^2(1-p^2x^2)(1-q^2x^2)$, let $px = y$, $q = pc$.
4. If $Q = m^2(1+p^2x^2)(1-q^2x^2)$, let
$$x^{-2} = (p^2+q^2)\,y^{-2} - p^2, \text{ and } p^2 = (p^2+q^2)\,c^2.$$
5. If $Q = m^2(1+p^2x^2)(x^2-y^2)$, let $qx^{-1} = \sqrt{(1-y^2)}$, $1 + p^2q^2 = c^{-2}$.
6. If $Q = m^2(p^2-x^2)(x^2-q^2)$, let $1 - q^2x^{-2} = c^2y^2$, $p^2 - q^2 = p^2c^2$.

[Whenever x is not rational in y, you must take care to get the true surd parts.]

4. Henceforward then, without loss of generality, we may assume $Q = (1-x^2)(1-c^2x^2)$ and $c^2 < 1$, and our problem is $\int \frac{R\,dx}{\sqrt{Q}}$ with R *rational* and *even* as a function of x. Then R has the form

$$\Sigma\alpha x^{2n} + \Sigma\beta x^{-2r} + \Sigma\frac{r}{(1+px^2)^m},$$

or three separate problems.

First, to find $U_n = \int_0 \frac{x^{2n}\,dx}{\sqrt{Q}}$, assume $V_n = x^{2n-3}\sqrt{Q}$. Differentiate the last; then, integrate back. It gives

$$V_n = (2n-3)\,U_{n-2} - (2n-2)(1+c^2)\,U_{n-1} + (2n-1)\,c^2U_n.$$

Hereby, U_n is reducible to U_{n-1} and U_{n-2}; and so, working back, we at length reach U_1 and U_0 or $\int_0 \frac{x^2\,dx}{\sqrt{Q}}$ and $\int_0 \frac{dx}{\sqrt{Q}}$.

Next, to find $\int \Sigma\beta\frac{x^{-2r}\,dx}{\sqrt{Q}}$, or terms of the type $\int \frac{x^{-2r}\,dx}{\sqrt{Q}}$ which we may call U_{-r}, we work by the last formula back from U_{-r} to U_{1-r} and U_{2-r}; and since none of the coefficients can now vanish, we reach U_0 and U_1 as before. For U_1 we may substitute $U_0 - c^2U_1$ or

$$\int_0 \sqrt{\frac{1-c^2x^2}{1-x^2}}\,.\,dx.$$

5. There remains the type

$$\int \frac{dx}{\sqrt{Q}\,.\,(1+px^2)^m} = S_m.$$

Take a larger assumption $Q = \alpha + \beta x^2 + \gamma x^4$, then by making

$$\alpha = 1, \ \beta = -(1+c^2), \ \gamma = c^2,$$

we regain the previous special value of Q.

Put $T_m = x\sqrt{Q}\,.\,(1+px^2)^m$. Differentiate T_{r-1};

$$\therefore\ dT_{r-1} = \sqrt{Q}^{-1}\,.\,(1+px^2)^{-r}.\,\{(1+px^2)\,Q + 2\,(1-r)\,px^2Q$$
$$+ (1+px^2)(\beta x^2 + 2\gamma x^4\}\,dx.$$

Assume as integral,

$$T_{r-1} = A\,.\,S_r + B\,.\,S_{r-1} + C\,.\,S_{r-2} + D\,.\,S_{r-3}.$$

Differentiate the last and compare it with the penultimate:

hence
$$\begin{aligned} A &= (2r-2)(\alpha - \ \beta p^{-1} + \ \gamma p^{-2}); \\ -B &= (2r-3)(\alpha - 2\beta p^{-1} + 3\gamma p^{-2}); \\ C &= (2r-4)(-\beta p^{-1} + 3\gamma p^{-2}); \\ +D &= (2r-5)(\gamma p^{-2}). \end{aligned}$$

Consequently S_r is reducible to $S_{r-1}S_{r-2}S_{r-3}$, and ultimately to $S_2S_1S_0$. In fact, if $r=2$, $C=0$, $\therefore$ S_2 is reducible to S_1 and S_{-1}. Now S_0 is the same thing as U_0 of last Article, when $\alpha\,\beta\,\gamma$ are fitly assumed;

and
$$S_{-1} = \int (1+px^2)\frac{dx}{\sqrt{Q}} = U_0 + pU_1.$$

Thus the only new and irreducible integral is S_1, or

$$\int \frac{dx}{(1+px^2)\sqrt{Q}} \text{ or } \int_0 \frac{d\omega}{(1+p\sin^2\omega)\,\Delta\,(c,\omega)},$$

if $x = \sin\omega$. [But x is better changed to v.]

6. Such is our Third Elliptic Integral, which remains to be treated. We denote it by $\Pi\,(c,\omega,p)$. In the reduction of a rational fraction $\dfrac{\phi(y)}{f(y)}$ to the form $\psi(y) + \Sigma\,.\dfrac{A}{1+py}$, p is not always real, but may be of the form $m + n\sqrt{-1}$. Such is the case here of p, which is called the *parameter* of the Third integral, which Legendre named the *Paranome.*

The case of p imaginary he beautifully reduces to two real parameters, as will presently appear. Evidently, when p is positive, the integral Π is always finite; so too, even when p is negative but numerically less than 1. In either case the integral is *periodic*, as was proved in Ch. I. Art. 3. But if p is negative and numerically exceeds 1, the integral becomes infinite when $1 + px^2 = 0$, i.e. *before*

ω reaches $\frac{1}{2}\pi$. We already see how important to our result will be the character of the parameter. A *positive* parameter we shall generally denote by $p=\cot^2\psi$, where ψ is a circular arc; then

$$1+p=\operatorname{cosec}^2\psi.$$

A *negative* parameter which is less than 1 might be simply $-\sin^2\alpha$, but it will soon appear that to be numerically less not than 1 only, but also than c^2 is important. To have the form $p'=-c^2\sin^2\theta$ is a cardinal fact; then $1+p'=\Delta^2(c\theta)$. Of course the arcs ψ and θ are either absolutely constant, or in any case independent of our variable ω. If $\sin\theta=\sqrt{-1}\tan\psi$, p' changes to $c^2\tan^2\psi$, *sub*conjugate to $\cot^2\psi$ or p.

7. The Complete Paranome is estimated when $\omega=\frac{1}{2}\pi$, and contains two elements only; viz., c the Modulus and p the Parameter. As our first effort was to determine F_c and E_c, so here our first effort must be to determine $\Pi_c(p)$. Happily the skill and industry of Legendre has reduced $\Pi_c(p)$, when finite, to known functions of $F(c\omega)$ and $E(c\omega)$, or rather of $G(c\omega)$.

When p is positive, evidently $\Pi(c\omega p)$ is less than $F(c\omega)$. The difference $F(c\omega)-\Pi(c\omega p)$ is in some formulas prominent. It

$$=\int_0 \frac{p\sin^2\omega}{1+p\sin^2\omega}\cdot\frac{d\omega}{\Delta(c\omega)},$$

and vanishes with p. It may be convenient to denote it by $\Phi(c\omega p)$ and its complete integral by $\Phi_c(p)$. It hardly needs a name. Of course $\Pi(c\omega p)=F(c\omega)-\Phi(c\omega p)$. Presuming F to be known, Π and Φ are known, each from the other.

It may be well to begin by *reducing to two elements* the single class of Paranomes which admit of this, viz. those which have a parameter of the form $p=-c^2\sin^2\theta$. With modulus ϵ, let

$$F\omega+F\theta=F\eta,\ F\omega-F\theta=F\kappa,$$

then (Ch. III. Art. 11),

$$\left.\begin{aligned}&G\omega+G\theta-G\eta=c^2\sin\omega\sin\theta\sin\eta\\&G\omega-G\theta-G\kappa=-c^2\sin\omega\sin\theta\sin\kappa\end{aligned}\right\},$$

whence $\qquad 2G\theta-(G\eta-G\kappa)=c^2\sin\omega\sin\theta(\sin\eta+\sin\kappa).$

But (Ch. III. Art. 3)

$$\sin\eta+\sin\kappa=\frac{2\sin\omega\cos\theta\,\Delta\theta}{1-c^2\sin^2\omega\sin^2\theta}.$$

Let θ be *constant;* $\qquad \therefore dF\eta=dF\omega=dF\kappa.$

Multiply by these and integrate,

$$\therefore\ 2G\theta\,.\,F\omega = \int_0 G\eta\,.\,dF\eta + \int_0 G\kappa\,.\,dF\kappa$$
$$= 2\sin\theta\cos\theta\,\Delta\theta\int_0 \frac{c^2\sin^2\omega\,dF\omega}{1-c^2\sin^2\theta\sin^2\omega}.$$

Otherwise

$$2G\theta\,.\,F\omega - \urcorner\eta + \urcorner\kappa = 2\cot\theta\,.\,\Delta\theta\int_0 \frac{-p\sin^2\omega}{1+p\sin^2\theta}\,.\,\frac{d\omega}{\Delta},$$
$$G\theta\,.\,F\omega - \tfrac{1}{2}(\urcorner\eta - \urcorner\kappa) = -\cot\theta\,.\,\Delta\theta\,.\,\Phi\,(c\omega p).$$

This reduces Φ, a function of three elements, to G, F and $\urcorner$, each of two elements: but only when p has the form $-c^2\sin^2\theta$, or p is negative and numerically less than c^2.

No constant of integration was needed: for F, $(\urcorner\eta - \urcorner\kappa)$, Φ all vanish with ω.

COR. When $\omega = \tfrac{1}{2}\pi,\ \eta + \kappa = \tfrac{1}{2}\pi,\ \urcorner\kappa = \urcorner\eta,$

$$\therefore\ \cot\theta\,.\,\Delta\theta\,.\,\Phi_c(p) = -G\theta\,.\,F_c.$$

8. We proceed to Legendre's reduction of the *Imaginary Parameter.*

Suppose c a given modulus, p and q given parameters,

$$Q = (1-v^2)(1-c^2v^2).$$

Write V for $\dfrac{v}{1+hv^2}\,.\,\sqrt{\dfrac{1-v^2}{1-c^2v^2}}$, where h is a disposable constant. Assume k for another such constant, and by differentiating V you find

$$\frac{dV}{1+kV^2} = \frac{1-(2+h)\,v^2+(1+2h)\,c^2v^4-hc^2v^6}{(1+hv^2)^2\,.\,(1-c^2v^2)+kv^2\,(1-v^2)}\,.\,dF.$$

Assume the denominator of the last fraction

$$= (1+pv^2)(1+qv^2)(1-rv^2),$$

where r is a disposable constant. This assumption implies three conditions; but we hold $h\,k\,r$ free to fulfil them. To obtain the three equations

$$\left.\begin{array}{l}\text{first, equate coefficients of } v^6;\ \text{then } pqr = h^2c^2,\\ \text{next, make } v^2 = 1;\\ \qquad\therefore\ (1+p)(1+q)(1-r) = (1+h)^2\,.\,b^2,\\ \text{thirdly, make } c^2v^2 = 1;\\ \qquad\therefore\ (c^2+p)(c^2+q)(r-c^2) = kb^2c^2,\end{array}\right\}\ (1).$$

To solve for $h\ r\ k$, first eliminate r from the two first of this triplet. It gives

$$(p^{-1}+1)(q^{-1}+1)(pq-h^2c^2)=(1+h)^2b^2.$$

Arrange this last in powers of h, as $Ah^2+2b^2h=B$, which yields by solving for h, $\quad Ah+b^2=\pm\sqrt{(AB+b^4)}$.

Here A stands for $\quad (p^{-1}+1)(q^{-1}+1)\,c^2+b^2$,

and B for $\quad (1+p)(1+q)-b^2$.

Then $AB=(p^{-1}+1)(q^{-1}+1)(1+p)(1+q)^2c^2$

$$-(p^{-1}+1)(q^{-1}+1)\,c^2b^2+(1+p)(1+q)\,b^2-b^4,$$

and $AB+b^4=(1+p)(1+q)\left\{(p^{-1}+1)(q^{-1}+1)\,c^2+b^2\left[1-\dfrac{c^2}{pq}\right]\right\}$.

But since $b^2=1-c^2$, the quantity in brackets

$$=1+(p^{-1}+q^{-1})\,c^2+c^4p^{-1}q^{-1} \text{ or } (1+c^2p^{-1})(1+c^2q^{-1}).$$

Finally, $AB+b^4=(1+p)(1+q)(1+c^2p^{-1})(1+c^2q^{-1})$.

This cannot vanish, unless either p or q were $=-1$ or $-c^2$. Each would be a case of degeneracy, but is here excluded because we are about to assign to p and q imaginary forms. Since then $AB+b^4$ cannot be zero, h has two *different* finite values (say h and h') from the equation $Ah+b^2=\pm\sqrt{(AB+b^4)}$. Then also two values of r follow from $pqr=h^2c^2$, and finally two values of k from the third equation of the triplet (1).

9. When $p=\alpha+\beta\sqrt{-1}$ and $q=\alpha-\beta\sqrt{-1}$, *I say*, $AB+b^4$ is *positive*, making both values of h real. For let

$$\alpha=\rho\cos\mu,\quad \beta=\rho\sin\mu,$$

$$\therefore\ p+q=2\rho\cos\mu,\quad pq=\alpha^2+\beta^2=\rho^2;$$

$$(1+p)(1+q)=1+2\rho\cos\mu+\rho^2,$$

positive. Also $\quad p^{-1}=\rho^{-1}(\cos\mu-\sqrt{-1}\sin\mu)$,

$$q^{-1}=\rho^{-1}(\cos\mu+\sqrt{-1}\sin\mu),\quad p^{-1}+q^{-1}=2\rho^{-1}\cos\mu,$$

and $p^{-1}q^{-1}=\rho^{-2}$,

$$\therefore\ (1+c^2p^{-1})(1+c^2q^{-1})=1+2c^2\rho^{-1}\cos\mu+c^4\rho^{-2},$$

again positive. Hence the product $AB+b^4$ is positive, as was asserted.

10. We may now write

$$\frac{dV}{1+kV^2} = \left\{H + \frac{L}{1+pv^2} + \frac{M}{1+qv^2} + \frac{N}{1-rv^2}\right\} \frac{dv}{\sqrt{Q}},$$

$$\frac{dV}{1+k'V^2} = \left\{H' + \frac{L'}{1+pv^2} + \frac{M'}{1+qv^2} + \frac{N'}{1-r'v^2}\right\} \frac{dv}{\sqrt{Q}},$$

where $HLMNH'L'M'N'$ are constants to be determined by ordinary routine, from the great fraction in Art. 8. Write $f(v^2)$ for its numerator $\quad 1-(2+h)v^2+(1+2h)c^2v^4-hc^2v^6.$

We have (when $v=\infty$),

$$H = \frac{-hc^2v^6}{-h^2c^2v^2}, \text{ or } H = \frac{1}{h},$$

and similarly $$H' = \frac{1}{h'}.$$

Next $$L = \frac{f(v^2)}{(1+qv^2)(1-rv^2)},$$

taken when $v^2 = -p^{-1}$; that is

$$L = \frac{f(p^{-1})}{(1-qp^{-1})(1-rp^{-1})};$$

or clearing fractions

$$L = \frac{p^3+(2+h)p^2+(1+2h)c^2p+hc^2}{p(p-q)(p+r)}.$$

From L you get M by exchanging p and q. Exchange p with $-r$, and you get N. Change h to h', and from LMN you find $L'M'N'$.

Since $p = \alpha + \beta\sqrt{-1}$, $\frac{L'}{L}$ must have the form $I + J\sqrt{-1}$. Change β to $-\beta$ which changes p to q; then

$$\frac{M'}{M} = I - J\sqrt{-1}.$$

So long as β is not zero, these two results are unequal, or $\frac{L'}{L}$ is *not* $=\frac{M'}{M}$.

Write $$K = \int_0 \frac{dV}{1+kV^2}, \quad K' = \int_0 \frac{dV}{1+k'V^2},$$

known integrals. Integrate with $v = \sin\omega$, the two equations obtained above: then

$$\left.\begin{array}{l} H \, . \, F(c\omega) + L \, . \, \Pi\,(c\omega p) + M\Pi\,(c\omega q) + N\Pi\,(c, \omega, -r) = K \\ H' . \, F(c\omega) + L' . \, \Pi\,(c\omega p) + M'\Pi\,(c\omega q) + N'\Pi\,(c, \omega, -r') = K' \end{array}\right\}.$$

Here we treat $\Pi\,(c\omega p)$, $\Pi\,(c\omega q)$, as two unknown quantities. We determine them by means of the two r Paranomi, in which each r is real, being $= \dfrac{h^2c^2}{pq} = \dfrac{h^2c^2}{\rho^2}$; and since $L : L'$ is *not* $M : M'$, this process never fails. Nothing imaginary enters

$$F(c\omega),\ \ \Pi\,(c\omega r)\ \text{ or }\ \Pi\,(c, \omega, -r').$$

11. *Species of the two new Parameters*, $-r$ and $-r'$. Note first that $1-r$, $1-r'$ are both positive. For

$$(1+h)^2 b^2 = (1+p)(1+q)(1-r).$$

We have shown that $(1+p)(1+q)$ is positive; so then is $(1-r)$, whichever root of h be taken. Next, *I say*, $r - c^2$ and $r' - c^2$ are of opposite signs. For $h\,h'$ being real roots of $Ah^2 + 2b^2h - B = 0$, make $Ahh' = -B$, $A\,(h+h') = -2b^2$. Now from $pqr = h^2c^2$ we get

$$pq\,(r-c^2) = (h^2 - pq)\,c^2, \quad pq\,(r'-c^2) = (h'^2 - pq)\,c^2;$$

$$\therefore\ p^2q^2\,(r-c^2)(r'-c^2) = (h^2-pq)(h'^2-pq)\,c^4$$

$$= (h^2h'^2 - pq\,.\,\overline{h^2+h'^2} + p^2q^2)\,c^4,$$

which again $\qquad = \{(hh'+pq)^2 - (h+h')^2 pq\}\,c^4.$

Multiply by A^2 and the right hand becomes

$$\{(-B + Apq)^2 - 4b^4pq\}\,c^4.$$

Now $\quad B = (1+p)(1+q) - b^2,\ \ Apq = (1+p)(1+q)\,c^2 + b^2pq,$

$$\therefore\ -B + Apq = -b^2\,(1+p)(1+q) + b^2\,(1+pq) = -b^2\,(p+q).$$

Hereby we find

$$A^2p^2q^2\,(r-c^2)(r'-c^2) = \{b^4\,(p+q)^2 - 4b^4pq\}\,c^4$$

$$= b^4c^4\,(p-q)^2 = -\,b^4c^4\,.\,4\rho^2\sin^2\mu,$$

a negative quantity. Hence $r - c^2$ and $r' - c^2$ are of opposite signs.

Thus $-r$ and $-r'$ both negative, are numerically less than 1, and lie, the one between 0 and $-c^2$, the other between $-c^2$ and -1.

In the third equation of condition (Art. 8) k has the same sign as $r - c^2$. Hence k and k' have opposite signs. Therefore of the integrals KK' one is found by circular arcs, the other by logarithms.

12. *Reciprocal Parameters.* It was not *necessary* in Art. 8 to assume p and q imaginary. If p alone were given, we might make one arbitrary assumption, having only three equations to satisfy with $qrhk$. Assume then $h=-1$, $1+h=0$.

This makes $r=1$, $pq=c^2$, a matter of cardinal importance. We then call p and q *Reciprocal Parameters.* Hence

$$(c^2+p)(c^2+q)=kc^2$$

in the third equation since $r-c^2=1-c^2=b^2$. Also $kc^2=kpq$,

$$\therefore\ hpq=(c^2+p)(c^2+q),$$

and $$k=(c^2p^{-1}+1)(c^2q^{-1}+1)=(q+1)(p+1).$$

The differential equation now becomes

$$\frac{dV}{1+kV^2}=\frac{1-c^2v^4}{(1+pv^2)(1+qv^2)}\cdot\frac{dv}{\sqrt{Q}},$$

since the numerator (when $h=-1$) becomes $1-v^2-c^2v^4+c^2v^6$ and has $(1-v^2)$ as a factor, which is cancelled now by $1-rv^2$ in the denominator, since $r=1$. Also

$$\frac{1-pqv^4}{(1+pv^2)(1+qv^2)}=\frac{1}{1+pv^2}+\frac{1}{1+qv^2}-1.$$

Hence by integration,

$$\int_0\frac{dV}{1+kV^2}=\Pi\,(c\omega p)+\Pi\,(c\omega q)-F\,(c\omega),$$

where $$k=(1+p)(1+q),$$

and $$pq=c^2,\ V=\frac{\tan\omega}{\Delta\,(c\omega)}\text{ since }h=-1.$$

The integration on the left varies with the sign of k.

Cor. If we make $p=q$, we have either $p=c$ or $p=-c$. In either case we obtain the value of the Paranome: that is, both of $\Pi\,(c\omega c)$ and of $\Pi\,(c,\omega,-c)$.

13. *Conjugate Parameters.* In the last, instead of assuming $h=-1$, assume $q=0$. Then since

$$pqr=h^2c^2,\ h=0\text{ and }(1+p)\,(1-r)=b^2$$

in the 2nd equation of the triplet (1).

Parameters p, p_0 so related that $(1+p)\,(1+p_0)=b^2$ are called Conjugate. *One* or *both* must be negative, as here $p_0=-r$, for if both were positive, b^2 would exceed 1, and c become imaginary.

If $-r$ have the form $-c^2 \sin^2 \theta$,

$$1+p=\frac{b^2}{\Delta^2 (c\theta)}.$$

Here, if θ θ^0 are Conjugate arcs reformed to modulus c, we have

$$1+p=\Delta^2 (c\theta^0),$$

or

$$p=-c^2 \sin^2 \theta^0.$$

Thus p is negative as well as $-r$, and we see the propriety of calling the parameters Conjugate.

From

$$(1+p)(1-r)=b^2=1-c^2,$$

we have

$$p-r=pr-c^2.$$

Also since

$$1-r=\frac{b^2}{1+p},$$

$$r=1-\frac{b^2}{1+p}=\frac{c^2+p}{1+p}.$$

If $p=-r_0$, negative as well as $-r$, $1-r$ and $1-r_0$ must have the same sign. *When both are negative,* call them $-\cot^2\theta$ and $-\cot^2\psi$, $\therefore$ $\cot\theta \, . \cot\psi=b$, or $\psi=\theta^0$. Also

$$r=\operatorname{cosec}^2\theta, \quad r_0=\operatorname{cosec}^2\theta^0.$$

When both are positive, the equation

$$r=\frac{c^2-r_0}{1-r_0}$$

shows that r_0 is less than c^2, so that $-r_0$ has the form $-c^2\sin^2\theta$, the case already treated; which gives $-r=-c^2\sin^2\theta^0$. But if p is positive, its conjugate must be negative. Put $p=\cot^2\psi$,

$$\therefore \quad 1+p=\operatorname{cosec}^2\psi,$$

and

$$1-r=b^2\sin^2\psi,$$

$$r=\Delta^2(b\psi),$$

with analogy to conjugate arcs.

14. *Equation of Conjugate Paranomes.* Since we made $q=0$ in Art. 13, the equation in Art. 8 which determines k is

$$b^2k=(p+c^2)(r-c^2).$$

Otherwise

$$b^2k=pr-c^2(p-r)-c^4,$$

and since $pr-c^2=p-r$ (in last Art.)

$$b^2k=pr-c^2(pr-c^2)-c^4=(1-c^2)pr=b^2pr,$$

so that k now $=pr$. Further, put (for the case of $q=0$ and $h=0$)

$$\frac{dV}{1+kV^2} \text{ or } \frac{1-2v^2+c^2v^4}{(1+pv^2)(1-rv^2)}=H+\frac{L}{1+pv^2}+\frac{R}{1-rv^2}.$$

Make $v=\infty$, $\quad \therefore \; H=\dfrac{c^2}{-pr}$.

Again multiply by $1-rv^2$, then make $v^2=r^{-1}$; which gives

$$R=\frac{1-2r^{-1}+c^2r^{-2}}{1+pr^{-1}}.$$

But $\quad p=\dfrac{r-c^2}{1-r}$ [since $(1+p)(1-r)=b^2$],

$$\therefore \; 1+pr^{-1}=1+\frac{1-c^2r^{-1}}{1-r}=\frac{2-r-c^2r^{-1}}{1-r}$$

$$=\frac{-(r-2+c^2r^{-1})}{1-r}=+\frac{1-2r^{-1}+c^2r^{-2}}{1-r^{-1}};$$

so that $R=1-r^{-1}$, $\therefore$ also $L=1+p^{-1}$; by changing $-r$ to p.

Hence $$\frac{dV}{1+prV^2}=\left(\frac{1+p^{-1}}{1+pv^2}+\frac{1-r^{-1}}{1-rv^2}-\frac{c^2}{pr}\right)\frac{dv}{\sqrt{Q}};$$

in which $$V=v\sqrt{\frac{1-v^2}{1-c^2v^2}}=\sin\omega\sin\omega^0,$$

since $h=0$. Finally

$$\int_0\frac{dV}{1+pr.V^2}=\frac{1+p}{p}\,\Pi\,(c\,\omega\,p)-\frac{1-r}{r}\,\Pi\,(c,\omega,-r)-\frac{c^2}{pr}.F(c\omega).$$

15. *Addition of Paranomic Integrals.* An integral

$$\int_0\phi\,(\omega).dF\,(c\omega)$$

is here called Paranomic, if ϕ is a function rational in $\sin^2\omega$. If $f(\omega)$ be such an Integral, Legendre demonstrates as follows that

$$U=f(\omega)+f(\theta)-f(\eta)$$

is expressible by circular arcs or logarithms, when

$$F(c\omega)+F(c\theta)=F(c\eta).$$

Proof. Let η be constant. Write

$$r=\sin^2\omega+\sin^2\theta,\quad s=\sin\omega\sin\theta.$$

$$\therefore\;\sqrt{(r^2-4s^2)}=\sin^2\omega-\sin^2\theta.$$

Now $$dF\omega+dF\theta=0,\quad dF\theta=-dF\omega,$$

$$dU=df\omega+df\theta=(\phi\omega-\phi\theta).dF\omega.$$

But if $\phi\omega$ be rational in $\sin^2\omega$, which

$$=\tfrac{1}{2}r+\tfrac{1}{2}\sqrt{(r^2-4s^2)},$$

$\phi\omega$ is expressible by $\quad R+S\sqrt{(r^2-4s^2)}$

where R and S are rational in r and s. Now to exchange ω with θ does but alter the sign of $\sqrt{}$ before r^2-4s^2, whence

$$\phi\theta=R-S\sqrt{(r^2-4s^2)}$$

with the same values of R and S, and

$$\phi\omega-\phi\theta=2S\sqrt{(r^2-4s^2)}=2S.(\sin^2\omega-\sin^2\theta),$$

$$\therefore\ dU=2S.(\sin^2\omega-\sin^2\theta)\,dF\omega.$$

Now we had $c^2d(\sin\omega\sin\theta\sin\eta)$ in the addition of Epinomes (Ch. III. Art. 11) $=dE\omega+dE\theta$, or

$$c^2.\sin\eta.ds=\Delta\omega d\omega+\Delta\theta d\theta=\Delta^2\omega.dF\omega+\Delta^2\theta dF\theta$$
$$=(\Delta^2\omega-\Delta^2\theta)\,dF\omega=-c^2(\sin^2\omega-\sin^2\theta)\,dF\omega,$$

$$\therefore\ \sin\eta.ds=-(\sin^2\omega-\sin^2\theta)\,dF\omega,$$

whence $\quad dU=-2S.\sin\eta.ds,$

$$\therefore\ U=-2\sin\eta\int Sds.$$

But by Euler's Integrals Ch. III. Art. 2,

$$\cos\eta=\cos\omega\cos\theta-s.\Delta\eta,$$

$$\therefore\ (\cos\eta+s\Delta\eta)^2=\cos^2\omega\cos^2\theta=(1-\sin^2\omega)(1-\sin^2\theta)=1-r+s^2.$$

Hence $\quad r=\sin^2\eta-2s\cos\eta\Delta\eta+c^2s^2\sin^2\eta,$

and r is rational in s. But S is rational in r and s, therefore also in s alone. Consequently $\int Sds$ is obtainable by circular arcs or logarithms. Q. E. D.

16. To apply this to the Paranome, let

$$\phi(\omega)=(1+p\sin^2\omega)^{-1},$$

$$r^2-4s^2=z,\quad \sin^2\omega=\tfrac{1}{2}(r+\sqrt{z});$$

$$\therefore\ R+S\sqrt{z}=\phi(\omega)=\{1+\tfrac{1}{2}p(r+\sqrt{z})\}^{-1}=\frac{(1+\frac{1}{2}pr)-\frac{1}{2}p\sqrt{z}}{(1+\frac{1}{2}pr)^2-(\frac{1}{2}p\sqrt{z})^2}$$

whence $\quad S=\dfrac{-\frac{1}{2}p}{1+pr+\frac{1}{4}p^2(r^2-z)},$

where $\frac{1}{4}(r^2 - z)$ in denominator $= s^2$;

$$\therefore \quad dU = -2 \sin \eta \,.\int S ds = \frac{p \sin \eta \,.\, ds}{1 + pr + p^2 s^2}.$$

Let $$h = \sin \eta, \quad H = \cos \eta \,.\, \Delta\eta,$$

$$\therefore \quad r = \sin^2 \eta - 2s \cos \eta \Delta\eta + c^2 s^2 \sin^2 \eta$$

$$= h^2 - 2Hs + c^2 h^2 s^2.$$

Substitute this in dU,

$$\therefore \quad U = \int_0 \frac{phds}{(1 + ph^2) - 2Hps + (p^2 + pc^2h^2) s^2};$$

manifestly integrable.

17. Nevertheless we have two cases. First, let the last denominator have no real factor. In that case we may assume

$$\alpha = 1 + ph^2, \quad \beta = p^2 + pc^2 h^2,$$

and $$\alpha t^2 = \beta s^2, \quad \alpha t \cos \gamma = Hps.$$

Then $$\cos \gamma = \frac{Hp}{\alpha} \,.\, \frac{s}{t} = \frac{Hp}{\sqrt{(\alpha\beta)}},$$

$$\sin \gamma = \frac{hp \sqrt{T}}{\sqrt{(\alpha\beta)}},$$

if $$H^2 p^2 + h^2 p^2 T = \alpha\beta = (1 + ph^2)(p^2 + pc^2 h^2).$$

Also $$H^2 = \cos^2 \eta \,.\, \Delta^2 \eta = (1 - h^2)(1 - c^2 h^2).$$

Hence you get by routine,

$$T = 1 + p + c^2 p^{-1} + c^2 = (1 + p)(1 + c^2 p^{-1}) = (1 + p)(1 + q),$$

if $pq = c^2$, or $p\ q$ be Reciprocal Parameters.

We have now, since $$ds = \sqrt{\frac{\alpha}{\beta}} \,.\, dt,$$

$$U = \frac{ph}{\sqrt{(\alpha\beta)}} \,.\, \int_0 \frac{dt}{1 - 2t \cos \gamma + t^2}.$$

We had $$hp \sqrt{T} = \sqrt{(\alpha\beta)} \sin \gamma,$$

$$\therefore \quad \frac{hp}{\sqrt{(\alpha\beta)}} = \frac{\sin \gamma}{\sqrt{T}};$$

whence $$\sqrt{T} \,.\, U = \tan^{-1} \,.\, \left(\frac{t \,.\, \sin \gamma}{1 - t \cos \gamma}\right).$$

This completes the integration, but we wish to replace t and γ by the given quantities.

Now $$t\sin\gamma = \left(\sqrt{\frac{\beta}{\alpha}}\,.\,s\right)\,.\,\frac{hp\sqrt{T}}{\sqrt{(\alpha\beta)}} = \frac{shp\sqrt{T}}{\alpha};$$

$$1 - t\cos\gamma = 1 - \frac{Hps}{\alpha};$$

$$\therefore\quad \frac{t\sin\gamma}{1 - t\cos\gamma} = \frac{\sqrt{T}\,.\,hps}{\alpha - Hps}.$$

Again, we had $\cos\eta = \cos\omega\cos\theta - s\,.\,\Delta\eta$ in Art. 15. Multiply it by $p\cos\eta$,

$$\therefore\quad p(1 - h^2) = p\cos\omega\cos\theta\cos\eta - psH,$$

also
$$\alpha = 1 + ph^2,$$

so that
$$\begin{aligned}\alpha - Hps &= (1 + ph^2) + (p - ph^2) - p\cos\omega\cos\theta\cos\eta\\ &= (1 + p) - p\cos\omega\cos\theta\cos\eta.\end{aligned}$$

Finally $$\sqrt{T}\,.\,U = \tan^{-1}.\,\frac{\sqrt{T}\,.\,p\sin\omega\sin\theta\sin\eta}{1 + p(1 - \cos\omega\cos\theta\cos\eta)};$$

in which
$$U = \Pi\omega + \Pi\theta - \Pi\eta,$$

and
$$T = (1 + p)(1 + c^2p^{-1})$$

18. Next, in reverse, suppose the denominator closing Art. 16 to have real factors. Denote them by $(1 - gs)(1 - js)$. Then $p\ h$ in the numerator being free forms, we may assume M an unknown constant, and write

$$\begin{aligned}U &= M\,.\int_0\frac{(g - j)\,ds}{(1 - gs)(1 - js)} = M\,.\int\left[\frac{ds}{1 - gs} - \frac{ds}{1 - js}\right]\\ &= M\,.\log\frac{1 - js}{1 - gs}.\end{aligned}$$

Then
$$(1 + ph^2)(1 - \overline{g + j}\,.\,s + gjs^2)$$

must be identified with the previous denominator, and

$$\begin{aligned}&(1 + ph^2)(g + j) = 2Hp,\\ &(1 + ph^2)\,gj = p^2 + pc^2h^2,\\ &M\,.\,(1 + ph^2)(g - j) = ph;\end{aligned}$$

three equations to determine $g\ j\ M$. Remember that

$$H^2 = (1 - h^2)(1 - c^2h^2).$$

Put
$$q = c^2 p^{-1},$$
$$\therefore\ p^2 + pc^2h^2 = p^2(1+qh^2).$$

Then
$$(g+j)^2 = \frac{4p^2(1-h^2)(1-c^2h^2)}{(1+ph^2)^2}$$

and
$$gj = \frac{p^2(1+qh^2)}{1+ph^2}.$$

Observe that

$$(1+ph^2)(1+qh^2) - (1-h^2)(1-c^2h^2) = (1+p+q+c^2)h^2$$
$$= (1+p)(1+q)h^2 = Th^2.$$

Hence
$$g - j = \frac{2ph\sqrt{(-T)}}{1+ph^2};$$
$$M = \frac{1}{2\sqrt{-T}},$$

or
$$\sqrt{-T}\,.\,U = \tfrac{1}{2}\log\frac{1-j\sin\omega\sin\theta}{1-g\sin\omega\sin\theta}.$$

Since $g+j$ and $g-j$ are both known, g and j can be found in terms of p and h, or say $h\ p\ q$ and $\sqrt{-T}$. The only point of interest to us is that T must be *negative*, when the integral is found by logarithms, and *positive* for circular arcs.

$\sqrt{T}\,.\,U$ is obtained in the one case, $\sqrt{-T}\,.\,U$ in the other.

[To complete the work:

$$g = \frac{p(H+h\sqrt{-T})}{1+ph^2},$$
$$j = \frac{p(H-h\sqrt{-T})}{1+ph^2}.$$

Observe that the fraction after log becomes

$$\frac{(1+ph^2)-ps(H-h\sqrt{-T})}{(1+ph^2)-ps(H+h\sqrt{-T})} \text{ or } \frac{1+p(h^2-Hs)+phs\sqrt{-T}}{1+p(h^2-Hs)-phs\sqrt{-T}}.$$

Now by Euler's Integrals

$$\cos\eta = \cos\omega\cos\theta - \sin\omega\sin\theta\Delta\eta,$$
$$\therefore\ \cos\omega\cos\theta = \cos\eta + \Delta\eta\,.\,s,$$

whence $\cos\eta\cos\omega\cos\theta = \cos^2\eta + \cos\eta\,.\,\Delta\eta\,.\,s = 1 - h^2 + Hs,$

$$\therefore\ h^2 - Hs = 1 - \cos\eta\cos\omega\cos\theta.$$

Thus the final result is

$$\sqrt{-T}.\{\Pi\omega+\Pi\theta-\Pi\eta\}$$
$$=\tfrac{1}{2}\log\frac{1+p(1-\cos\omega\cos\theta\cos\eta)+p\sin\omega\sin\theta\sin\eta\sqrt{-T}}{1+p(1-\cos\omega\cos\theta\cos\eta)-p\sin\omega\sin\theta\sin\eta\sqrt{-T}}.$$

When $\eta=\tfrac{1}{2}\pi$, ω and θ are conjugate, $\cos\eta=0$, but the equation is still cumbrous.]

19. The function $T=(1+p)(1+c^2p^{-1})$ may be called DIACRITIC of parameters. It separates them into two classes. When T is *positive*, Legendre entitles the parameter *circular*, because its Paranome depends on circular arcs; when T is negative, he entitles the parameter logarithmic. When p is of the form

$$-c^2\sin^2\theta,\qquad 1+p=\Delta^2(c\theta),$$
$$c^2p^{-1}=-\operatorname{cosec}^2\theta,\quad 1+c^2p^{-1}=-\cot^2\theta,$$
$$\therefore\quad T\text{ or }T(p)=-\cot^2\theta\,.\,\Delta^2(c\theta),$$

a negative quantity.

Looking back to Art. 7 in which $\Phi(c\omega p)$ was reduced to ꞁ, G and F when $p=-c^2\sin^2\theta$, it will be observed that Φ is there multiplied by $\cot\theta\,.\,\Delta\theta$ or $\sqrt{-T}$, so that the integral reduced is virtually $\sqrt{-T}\,.\,\Phi$. It is, we see, a condition that T be negative.

If p be negative but numerically >1, we denote it by $p=-\operatorname{cosec}^2\theta$. In that case there is no *complete* function Π_c, for $1+p\sin^2\omega$ becomes zero and Π infinite, before ω reaches $\tfrac{1}{2}\pi$. But when $p=-\operatorname{cosec}^2\theta$, its reciprocal is $p'=-c^2\sin^2\theta$, and by the equation of Art. 12 $\Pi(p)$ can be found from $\Pi(p')$, which latter is reducible to ꞁ. Moreover in Art. 12 the k on the left hand is our Diacritic T, and to integrate $\int_0\frac{dV}{1+kV^2}$ you first multiply by $\sqrt{k}$, thus adding the factor $\sqrt{-T}$ before Π (since T is here negative). $T=(1+p)(1+q)$, and is the same for each Reciprocal parameter. Thus again we find $\sqrt{-T}\,.\,\Pi$ to be the function dealt with.

When p is positive, so is q. We may then put

$$p=\cot^2\psi,\quad q=c^2\tan^2\psi.$$

Also $T=\operatorname{cosec}^2\psi(1+c^2\tan^2\psi)$

$$=\frac{1}{\sin^2\psi}\cdot\frac{\cos^2\psi+c^2\sin^2\psi}{\cos^2\psi}=\frac{1}{\sin^2\psi}\cdot\frac{1-b^2\sin^2\psi}{\cos^2\psi}=\frac{1}{\sin^2\psi\sin^2\psi_0},$$

if ψ_0 be a *sub*conjugate arc to ψ; i.e. if $F(b\psi)+F(b\psi_0)=F_b$.

N.B. In this treatise arcs ω, ω^0, ω_0 are so distinguished:

$$F(c\omega)+F(c\omega^0)=F_c;\ F(b\omega)+F(b\omega_0)=F_b.$$

From $\sin\omega^0=\dfrac{\cos\omega}{\Delta(c\omega)}$ conjugate

we deduce $\sin\omega_0=\dfrac{\cos\omega}{\Delta(b\omega)}$ sub-conjugate.

The formula $\sqrt{T}=\dfrac{1}{\sin\psi\,.\sin\psi_0}$ is often convenient.

Finally, if p' is between $-c^2$ and -1, we may assume for it $-p'=1-b^2\sin^2\psi$, so that

$$1+p'=b^2\sin^2\psi,\ -q'=\frac{c^2}{\Delta^2(b\psi)}=\Delta^2(b\psi_0),\ 1+q'=b^2\sin^2\psi_0,$$

$\therefore\ T'=b^4\,.\sin^2\psi\,.\sin^2\psi_0$ positive, and $\sqrt{T'}=b^2\sin\psi\,.\sin\psi_0$.

It will be remarked that $\sqrt{T}\,.\sqrt{T'}$ here make b^2, when the p of T is $\cot^2\psi$, and the p' of T' is $-\Delta^2(b\psi)$.

20. If $p=\cot^2\psi$, $\quad q=c^2\tan^2\psi=\cot^2\psi_0$.

Further, by Art. 13, if $-r$ is conjugate to p,

$$r=\frac{c^2+p}{1+p}=\frac{c^2+c^2\tan^2\psi_0}{1+\cot^2\psi}=c^2\,.\sin^2\psi\,.\sec^2\psi_0=\frac{c^2\sin^2\psi}{\cos^2\psi_0}.$$

But $\cos\psi_0=\dfrac{c\sin\psi}{\Delta(b\psi)}$, $\therefore\ r=\Delta^2(b\psi)$.

It follows that if $-r'$ is conjugate to q, $\therefore\ r'=\Delta^2(b\psi_0)$. Also

$$\Delta(b\psi)\,.\Delta(b\psi_0)=c,$$

because $\psi\psi_0$ are subconjugate: hence $rr'=c^2$ and rr' are Reciprocals. Thus in *this* class of parameters we find the Conjugates of Reciprocals to be Reciprocals, and the Reciprocals of Conjugates to be Conjugate. But the same is easily proved true of the opposite class, in which

$$p=-c^2\sin^2\theta,\quad q=-\operatorname{cosec}^2\theta,$$

$$r=\frac{c^2+p}{1+p}=\frac{c^2-c^2\sin^2\theta}{1-c^2\sin^2\theta}=\frac{c^2\cos^2\theta}{\Delta^2(c\theta)}=c^2\sin^2\theta^0;$$

again $$r'=\frac{q+c^2}{q+1}=\frac{1-c^2\sin^2\theta}{1-\sin^2\theta}=\frac{\Delta^2(c\theta)}{\cos^2\theta}=\frac{1}{\sin^2\theta};$$

whence $rr'=c^2$, as before.

We had $\sqrt{-T}=\cot\theta.\Delta\theta$ from $p=-c^2\sin^2\theta$. From $p'=-c^2\sin^2\theta^0$ (conjugate to the former) we have similarly $\sqrt{-T'}=\cot\theta^0\,.\Delta\theta^0$.

Hence $$\sqrt{(TT')}=\cot\theta\,.\cot\theta^0\,.\Delta\theta\,.\Delta\theta^0.$$

But $\cot\theta\,.\cot\theta^0 = b$, and $\Delta\theta\,.\,\Delta\theta^0 = b$; $\therefore\ TT' = b^4$,
as in the other class of parameters.

The Diacritic so uniformly attends the Paranome Π as a factor prefixed, that it is expedient to regard $\sqrt{T}\,.\,\Pi$ as the function under treatment. I call it, the HYPERNOME. When T is negative, the Paranome is superseded by ℸ, and is merely an occasional Auxiliary. When $T\,.\,\Pi$ are written *together*, they are understood to have the *same* parameter.

21. In Article 7 above, having

$$p = -c^2\sin^2\theta \text{ and } T = -\cot\theta\,\Delta(c\theta),$$

we restore to Φ its value $F - \Pi$, and obtain

$$\sqrt{-T}\,.\,\{F(c\omega) - \Pi(c\omega p)\} = \tfrac{1}{2}(ℸ\eta - ℸ\kappa) - G\theta\,.\,F\omega.$$

Let $$\omega = \tfrac{1}{2}\pi.$$

Then for the Complete Hypernome,

$$\sqrt{-T}\{F_c - \Pi_c(p)\} = -G\theta\,.\,F_c;$$

Π_c now exceeding F_c ....(1).

This formula is now an aid to find Π_c when the parameter is circular. For since it has been attained by finite integration, it will not be vitiated by assigning to θ an imaginary form $\alpha \pm \beta\sqrt{-1}$.

22. *Complete Hypernome with positive parameter.* In the last make $\sin\theta = \sqrt{-1}\tan\psi_0$, then

$$p = -c^2\sin^2\theta = c^2\tan^2\psi_0 = \cot^2\psi.$$

We must pass from Nomiscus G to Antinomiscus J, as in Ch. II. § 6, Art. 2,

$$G(c\theta) = \sqrt{-1}\,\{\tan\psi_0\,\Delta(b\psi_0) - J(b\psi_0)\}.$$

For $\sqrt{-T}$ we have only to write $\pm\sqrt{-1}\,.\,\sqrt{T}$. Then we find

$$\pm\sqrt{-1}\,.\,\sqrt{T}\,\{F_c - \Pi_c(p)\} = -\sqrt{-1}\,\{\tan\psi_0\,.\,\Delta(b\psi_0) - J(b\psi_0)\}\,F_c.$$

When b is very small,

$$\Delta(b\psi_0) = 1 - \tfrac{1}{2}b^2\sin^2\psi_0, \text{ and } J(b\psi_0) = (1 - \tfrac{1}{2}b^2)\sin\psi_0,$$

while $\tan\psi_0$ is positive and may be of any magnitude. This relieves doubt as to $\pm$, and gives us, when $p = \cot^2\psi$,

$$\sqrt{T}\,.\,\{F_c - \Pi_c(p)\} = \{\tan\psi_0\,.\,\Delta(b\psi_0) - J(b\psi_0)\}\,F_c \text{ ...... (2).}$$

Now $$\tan\psi_0 = \frac{1}{c\tan\psi}, \quad \Delta(b\psi_0) = \frac{c}{\Delta(b\psi)},$$

$$\therefore \tan\psi_0 . \Delta(b\psi_0) = \frac{1}{\sqrt{(pr)}},$$

if r is conjugate to p.

But we naturally desire to have the right member in functions of ψ. From Ch. II. § 6, Art. 3,

$$J(b\psi) + J(b\psi_0) - J_b = b^2 \sin\psi \sin\psi_0.$$

Indeed $$J_c = \frac{\frac{1}{2}\pi}{F_b}, \quad \therefore J_b . F_c = \tfrac{1}{2}\pi.$$

First, eliminating $J(b\psi_0)$ from equation (2) the right hand becomes

$$\tan\psi_0 . \Delta(b\psi_0) . F_c + J(b\psi) F_c - \tfrac{1}{2}\pi - b^2 \sin\psi . \sin\psi_0 . F_c.$$

For a moment write

$$u = \tan\psi_0 . \Delta(b\psi_0) - b^2 \sin\psi \sin\psi_0.$$

Remove ψ_0. Then

$$u = \frac{\cot\psi}{c} . \frac{c}{\Delta(b\psi)} - b^2 \sin\psi . \frac{\cos\psi}{\Delta(b\psi)},$$

or $$u = \frac{\cot\psi}{\Delta(b\psi)} \{1 - b^2 \sin^2\psi\} = \cot\psi . \Delta(b\psi),$$

$$\therefore \sqrt{T} . \{F_c - \Pi_c(p)\} = \{\cot\psi . \Delta(b\psi) + J(b\psi)\} F_c - \tfrac{1}{2}\pi \ldots(3).$$

Again, since $$\sqrt{T} = \frac{1}{\sin\psi \sin\psi_0},$$

we can combine $\sqrt{T} - \tan\psi_0 . \Delta(b\psi_0)$ in equation (2). It becomes

$$\frac{\Delta(b\psi_0)}{\cos\psi_0} . \frac{1}{\sin\psi_0} - \frac{\Delta(b\psi_0)\sin\psi_0}{\cos\psi_0}$$

$$= \frac{\Delta(b\psi_0)}{\cos\psi_0 . \sin\psi_0} \{1 - \sin^2\psi_0\} = \Delta(b\psi_0) . \cot\psi_0,$$

$$\therefore \sqrt{T} . \Pi_c(p) = \{\Delta(b\psi_0)\cot\psi_0 + J(b\psi_0)\} . F_c \quad \ldots\ldots\ldots(4)$$

$$= \left\{\frac{c^2}{\sqrt{(pr)}} + J(b\psi_0)\right\} F_c,$$

if $-r$ is conjugate to p. To remember these formulas is arduous.

Unless b is near to 1, we may calculate $G(b\psi_0)$, with ρ' related to b_1 as ρ to c_1 and Mesonome y for G and ψ_0 from

$$\tfrac{1}{2} BG(b\psi_0) = \frac{\sin 2y}{\text{Sin } 2\rho'} + \frac{\sin 4y}{\text{Sin } 4\rho'} + \frac{\sin 6y}{\text{Sin } 6\rho'} + \&\text{c.} \quad [\text{Ch. VIII. Art. 8}],$$

and thence obtain

$$J(b\psi_0) = G(b\psi_0) + \frac{y}{F_c}. \qquad \text{[Ch. II. § 6, Art. 2.]}$$

Then, knowing $J(b\psi_0)$, we obtain

$$\sqrt{T}\,.\,\frac{\Pi_c(p)}{F_c} = \frac{c^2}{\sqrt{(pr)}} + J(b\psi_0).$$

If b is near to 1, we may calculate $BJ(b\psi_0)$ from the formula for $CJ(c\omega)$ in Ch. IV. § 9, Art. 5, Lagrange's scale used in reversed order. For another approximation to $C\,.\,J(b\psi_0)$ when b is large, see Art. 16 in close of Ch. VIII.

23. *Complete Hypernome* with parameter *circular* but *negative.* We must reduce this by the general equation of Conjugate Paranomes, in Art. 14. Multiply it by $\sqrt{(pr)}$. Observe that

$$\sqrt{pr}\,.\,\frac{1+p}{p} = \frac{1+p}{\sqrt{p}}\,.\,\sqrt{r} = \frac{1+p}{\sqrt{p}}\,.\,\sqrt{\frac{c^2+p}{1+p}}$$

$$= \sqrt{1+p}\,\sqrt{c^2p^{-1}+1} = \sqrt{T}\,.\,(p).$$

Similarly $$\sqrt{(pr)}\,\frac{1-r}{r} = \frac{1-r}{\sqrt{r}}\,\sqrt{p} = \frac{1-r}{\sqrt{r}}\,.\,\sqrt{\frac{1-c^2}{1-r}}$$

$$= \sqrt{(1-r)}\,\sqrt{1-c^2r^{-1}} = \sqrt{T}\,(-r).$$

Thus the equation of Conjugate *Hypernomes* becomes

$$\int_0 \frac{\sqrt{(pr)}\,.\,dV}{1+pr\,.\,V^2},$$

or $$\tan^{-1}(\sqrt{pr}\,.\,V) = \sqrt{T}\,.\,\Pi(c\omega p) - \sqrt{T}\Pi(c\omega - r) - \frac{c^2}{\sqrt{(pr)}}\,.\,F(c\omega),$$

in which $$V = \sin\omega\sin\omega^0.$$

Make $$\omega = \tfrac{1}{2}\pi, \quad \therefore\ \omega^0 = 0, \ V = 0,$$

$$\sqrt{T}\Pi_c(p) - \sqrt{T}\Pi_c(-r) = \frac{c^2}{\sqrt{(pr)}}\,.\,F_c.$$

But we had $$\sqrt{T}\,.\,\Pi_c(p) = \left\{\frac{c^2}{\sqrt{(pr)}} + J(b\psi_0)\right\}F_c,$$

$$\therefore\ \sqrt{T}\,.\,\Pi_c(-r) = J(b\psi_0)\,.\,F_c.$$

This result is unexpectedly simple.

24. *Reciprocal Hypernomes.* In Art. 12 the

$$k = (1+p)(1+q) = T \text{ and } V = \frac{\tan\omega}{\Delta(c\omega)}.$$

Multiply the Reciprocal equation by $\sqrt{T}$,

$$\therefore \tan^{-1}(\sqrt{T}.V)=\sqrt{T}.\Pi(c\omega p)+\sqrt{T}\Pi(c\omega q)-\sqrt{T}.F(c\omega).$$

Also when $\omega=\frac{1}{2}\pi$, $V=\infty$,

$$\tan^{-1}(\sqrt{T}.V)=\tan^{-1}(\infty)=\tfrac{1}{2}\pi,$$

so for the Complete Hypernomes

$$\sqrt{T}\{\Pi_c(p)+\Pi_c(q)-F_c\}=\tfrac{1}{2}\pi.$$

25. Complete Hypernomes may also be developed in series.

As a guide, take first the parameter $p=-c^2\sin^2\theta$, and from $c\theta$ form as in Lagrange's scale $c_1\theta_1c_2\theta_2c_3\theta_3\ldots$. Then we have

$$\sqrt{-T}.\{\Pi_c(p)-F_c\}=F_c.G(c\theta)=\tfrac{1}{2}\pi.CG(c\theta).$$

But we know that

$$CG(c\theta)=C_1c_1\sin\theta_1+C_2c_2\sin\theta_2+C_3c_3\sin 3\theta_3+\&c. \quad (\S\ 9,\ \text{Art. } 2).$$

Assume $\qquad p_n=-c_n{}^2\sin^2\theta_n,$

$$\therefore \sqrt{-T}\{\Pi_c(p)-F_c\}=\tfrac{1}{2}\pi.\{C_1\sqrt{-p_1}+C_2\sqrt{-p_2}+C_3\sqrt{-p_3}+\&c.\}.$$

Transform by putting $\sin\theta=\sqrt{-1}\tan\psi_0$. $\sqrt{-T}$ changes as before to $-\sqrt{-1}.\sqrt{T}$,

$$\therefore \sqrt{T}\{F_c-\Pi_c(p)\}=\tfrac{1}{2}\pi\{C_1\sqrt{p_1}+C_2\sqrt{p_2}+C_3\sqrt{p_3}+\&c.\}$$

for positive parameter.

But the law of $pp_1p_2p_3\ldots$ must now be calculated as in the scale of Gauss, and if our object is simply to develop in series, we need not concern ourselves with any but p.

26. Begin anew with $p=-c^2\sin^2\theta$. Let

$$M=\frac{1+p}{p}.\Phi_c(p) \text{ or } =\int_0^{\frac{1}{2}\pi}\frac{(1+p)\sin^2\omega}{1+p\sin^2\omega}.\frac{d\omega}{\Delta(c\omega)},$$

with $(1+p)$ positive. Then

$$pM=(1+p)\{F_c-\Pi_c(p)\},$$

and we had $\qquad \sqrt{-T}.(\Pi_c.p-F_c)=F_c.G\theta.$

Here $\qquad \sqrt{-T}=\cot\theta.\Delta(c\theta).$

Therefore $F_c.G(c\theta)=\sqrt{-T}.\left(\dfrac{-p}{1+p}\right).M=\cot\theta\Delta\theta.\left(\dfrac{c^2\sin^2\theta}{\Delta^2\theta}\right).M$

$$=\frac{c^2\sin\theta.\cos\theta}{\Delta\theta}.M=(1-b)\sin\theta_1.M.$$

When $c\theta$ changes to $c_1\theta_1$, let M become M_1, then with

$$\tfrac{1}{2}\pi C.G(c\theta)=(1-b)\sin\theta_1.M$$

you have $\qquad \dfrac{c_1{}^2\sin\theta_1\cos\theta_1}{\Delta(c_1\theta_1)}.M_1=\tfrac{1}{2}\pi C_1G(c_1\theta_1).$

We know that $\quad CG(c\theta) - C_1 G(c_1\theta_1) = C_1 c_1 \sin\theta_1.$

Hence $\quad (1-b)\sin\theta_1 . M - \dfrac{c_1^2 \sin\theta_1 \cos\theta_1}{\Delta(c_1\theta_1)} . M_1 = \tfrac{1}{2}\pi . C_1 c_1 \sin\theta_1.$

Divide by $\quad C(1-b)\sin\theta_1 = 2C_1 c_1 \sin\theta_1,$

$$\therefore \frac{M}{C} - \frac{c_1}{2} . \frac{\cos\theta_1}{\Delta(c_1\theta_1)} . \frac{M_1}{C_1} = \tfrac{1}{4}\pi.$$

We form the series $\lambda\lambda_1\lambda_2\lambda_3\ldots$ from p through θ from

$$\sin\lambda = \frac{b}{\Delta(c\theta)},$$

which is to entail $\quad \sin\lambda_n = \dfrac{b_n}{\Delta(c_n\theta_n)}.$

Then $\quad \cos\lambda = \dfrac{c\cos\theta}{\Delta(c\theta)}, \quad \cos\lambda_n = \dfrac{c_n\cos\theta_n}{\Delta(c_n\theta_n)}.$

Now in Lagrange's scale

$$(1+b)\,c_1\cos\theta_1 = \Delta - \Delta^0 \text{ and } (1+b)\,\Delta_1 = \Delta + \Delta^0,$$

$$\therefore \frac{c_1\cos\theta_1}{\Delta_1} = \frac{\Delta - \Delta^0}{\Delta + \Delta^0}.$$

But we had $\quad \cos\lambda_1 = \dfrac{c_1\cos\theta_1}{\Delta_1}.$

Hence $\quad \cos\lambda_1 = \dfrac{\Delta - \Delta^0}{\Delta + \Delta^0}.$

From this again $\quad \dfrac{1-\cos\lambda_1}{1+\cos\lambda_1} = \dfrac{\Delta^0}{\Delta} = \dfrac{b}{\Delta^2},$

or $\quad \tan\tfrac{1}{2}\lambda_1 = \dfrac{\sqrt{b}}{\Delta}.$

But $\quad \sin\lambda = \dfrac{b}{\Delta}, \quad \therefore \tan\tfrac{1}{2}\lambda_1 = \dfrac{\sin\lambda}{\sqrt{b}}.$

I presume that this beautiful adaptation is from the genius of Legendre.

We have now $\quad \dfrac{M}{C} = \tfrac{1}{4}\pi + \tfrac{1}{2}\cos\lambda_1 . \dfrac{M_1}{C_1},$

as an equation of reduction. It at once gives

$$\frac{M}{C} = \tfrac{1}{4}\pi + \tfrac{1}{2}\cos\lambda_1 \{\tfrac{1}{4}\pi + \tfrac{1}{2}\cos\lambda_2 \{\tfrac{1}{4}\pi + \tfrac{1}{2}\cos\lambda_3 \{\tfrac{1}{4}\pi + \&c.,$$

or $M = \tfrac{1}{4}\pi C . \{1 + \tfrac{1}{2}\cos\lambda_1 + \tfrac{1}{4}\cos\lambda_1\cos\lambda_2 + \tfrac{1}{8}\cos\lambda_1\cos\lambda_2 + \cos\lambda_3 + \&c.$

27. The last has been deduced from the hypothesis $p = -c^2 \sin^2\theta$. But, provided that we rightly determine λ, it will hold with a positive parameter. Instead of $\sin\lambda = \dfrac{b}{\Delta(c\theta)}$,

where $\Delta(c\theta)$ or $\sqrt{(1-c^2\sin^2\theta)} = \sqrt{(1+p)}$,

we have only to take $\sin\lambda = \dfrac{b}{\sqrt{(1+p)}}$ and then form the series $\lambda_1\lambda_2\lambda_3\ldots$ by the same law as before, viz. $\tan\frac{1}{2}\lambda_1 = \dfrac{\sin\lambda}{\sqrt{b}}$. Since the series $bb_1b_2\ldots$ rapidly tends to 1, the series $\lambda\lambda_1\lambda_2\ldots$ tends to 90°, as $\cos\lambda_1$ $\cos\lambda_2$ rapidly diminish. If b were actually 1, we might have

$$\lambda = 90^\circ, \ \lambda_1 = 90^\circ.$$

If $$p = \cot^2\psi, \ \sin\lambda = b\sin\psi.$$

Also $$\Pi_c(p) - F_c = \int_0^{\frac{1}{2}\pi} \frac{p\sin^2\theta}{1+p\sin^2\theta}\cdot\frac{d\omega}{\Delta(c\omega)} = \frac{p}{1+p}\cdot M.$$

Thence $$\Pi_c p = F_c + \frac{p}{1+p}\cdot M$$

$$= \tfrac{1}{2}\pi C\left\{1 + \frac{\frac{1}{2}p}{1+p}(1 + \tfrac{1}{2}\cos\lambda_1 + \tfrac{1}{4}\cos\lambda_1 . \cos\lambda_2 + \tfrac{1}{8} . \&c.)\right\}.$$

28. This seems to close all that can be attained in the Paranome while our formulas ultimately exact two elements only. While ω remains indefinite, and the parameter is of the circular class, the Paranome remains a function of three independent elements. But the fluctuating part by which it exceeds the Complete Paranome $\Pi_c(p)$ may be reduced to a small and intelligible state.

CHAPTER XI.

PARANOME.

Second Part.

THREE INDEPENDENT ELEMENTS.

1. *Paranomiscus.* As we invented the Nomiscus as a *fluctuating* element by which the Epinome differs from a multiple of E_c, similarly we devise a function $P(c\omega p)$ as Paranomiscus, which we define by $P = \Pi - \frac{\Pi_c}{F_c} . F$. This evidently vanishes at every complete quadrant: for if $\omega = n . \frac{1}{2}\pi$, $F = n . F_c$ and $\Pi = n . \Pi_c$,

$$\therefore \; P = n . \Pi_c . - \frac{\Pi_c}{F_c} . nF_c = \text{zero}.$$

Knowing P we shall deduce the indefinite integral Π from $P + \frac{\Pi_c}{F_c} . F$; and it is presumed that the fluctuating function P, lying within narrow limits, will be easier of calculation than Π. It will follow, to call $\sqrt{TP}$ the Hypernomiscus. Evidently

$$P(n\pi + \omega) = P(\omega),$$

whatever c and p may be.

2. *Case of* p *infinitesimal.* When $p = 0$, $\Pi = F$ and $\Pi_c = F_c$. Evidently therefore P vanishes with p. But this is not obvious concerning the *Hyper*nomiscus $\sqrt{T\Pi}$; for from $p = 0$, we deduce T infinite and Π zero. In fact $T = (1 + p)\left(1 + \frac{c^2}{p}\right)$ which converges to

$$\left(1 + \frac{c^2}{p} + c^2\right).$$

Observe that when p is very small

$$(1+p\sin^2\omega)^{-1} = 1 - p\sin^2\omega,$$

$$\therefore\ \Pi = \int_0 (1-p\sin^2\omega)\frac{d\omega}{\Delta(c\omega)} = F - p\int_0 \frac{\sin^2\omega\, d\omega}{\Delta(c\omega)}$$

$$= F - \frac{p}{c^2}(F-E) = \left(1-\frac{p}{c^2}\right)F + \frac{p}{c^2}\,.\,E.$$

Make $\omega = \frac{1}{2}\pi,\ \therefore\ \Pi_c = \left(1-\frac{p}{c^2}\right)F_c + \frac{p}{c^2}\,.\,E_c.$

Multiply this by $\dfrac{F}{F_c}$ and subtract from the penultimate,

$$\therefore\ \Pi - \Pi_c\,.\frac{F}{F_c} = \frac{p}{c^2}\left(E - E_c\,.\,\frac{F}{F_c}\right),\ \text{i.e. } P = \frac{p}{c^2}\,.\,G,$$

when p is infinitesimal. Multiply by

$$\sqrt{T} = \sqrt{\overline{1+p}\,.\,\overline{1+c'^2p^{-1}}}$$

then $$\sqrt{TP} = \sqrt{\overline{1+p}\,.\,\overline{p+c^2}}\,.\,\frac{\sqrt{p}}{c^2}\,.\,G,$$

which vanishes with $\sqrt{p}$ though more slowly than does P. Herein, p may be of either sign.

3. *Case of* p *positive* and *infinite.* Evidently $(1+p\sin^2\omega)^{-1}$ then vanishes; so then does Π and P. But $\sqrt{T}$ being then infinite, further inquiry is needed concerning $\sqrt{T\Pi}$ and $\sqrt{TP}$. Since

$$(1+p\sin^2\omega)^{-1}$$

is zero, when p is positive and infinite the integral receives no accession after ω has become sensible. When ω is infinitesimal, $\sin\omega = \omega$,

and $$\Delta(c\omega) = 1;\ (1+p)(1+c^2p^{-1}) \text{ is } = 1+p = p,$$

therefore $$\sqrt{T\Pi} = \int_0 \sqrt{p}\,.\,\frac{d\omega}{1+p\omega^2} = \tan^{-1}(\sqrt{p}\omega),$$

which as soon as ω is sensible, becomes $\tan^{-1}(\infty) = \frac{1}{2}\pi$, if p is positive and infinite. Thus $\sqrt{T\Pi}\,(c\omega p) = \frac{1}{2}\pi$, whatever the finite value of ω and c.

Since then also $\sqrt{T\Pi_c} = \frac{1}{2}\pi$, it follows that

$$\sqrt{T}\,.\,P\,(c\omega p) = \frac{1}{2}\pi\left(1 - \frac{F(c\omega)}{F_c}\right),$$

when p is positive and infinite.

4. *Reciprocal Hypernomisci.* In Art. 24 of last Chapter we had for Reciprocal Hypernomes

$$\sqrt{T}\Pi(c\omega p) + \sqrt{T}\Pi(c\omega q) - \sqrt{T}\,.\,F(c\omega) = \tan^{-1}\left(\sqrt{T}\,.\,\frac{\tan\omega}{\Delta(c\omega)}\right) = (\text{say})\ \xi,$$

and when $\omega = \frac{1}{2}\pi$,

$$\sqrt{T}\Pi_c(p) + \sqrt{T}\Pi_c(q) - \sqrt{T}\,.\,F_c = \tfrac{1}{2}\pi.$$

$$\left(\text{Observe that } \tfrac{1}{2}\pi\,.\,\frac{F}{F_c} = \text{mesonome } x.\right)$$

Multiply the last by $\frac{F}{F_c}$ and subtract from the penultimate; then

$$\sqrt{T}P(c\omega p) + \sqrt{T}P(c\omega q) = \xi - x.$$

Here we have $$\tan\xi = \sqrt{T}\,.\,\frac{\tan\omega}{\Delta(c\omega)}.$$

But if $$p = \cot^2\psi, \quad q = c^2\tan^2\psi,$$

$$\sqrt{T} = \frac{1}{\sin\psi\,.\,\sin\psi_0}.$$

Also $$\frac{\tan\omega}{\Delta(c\omega)} = \frac{\sin\omega}{\cos\omega\,\Delta(c\omega)} = \frac{\cos\omega^0}{b\cos\omega}.$$

Thus $$\tan\xi = \frac{1}{\sin\psi\sin\psi_0}\,.\,\frac{\cos\omega^0}{b\cos\omega}.$$

With $r = \Delta^2(b\psi)$, $-r$ will be the parameter conjugate to p. Write Ω for $\sqrt{T}P(c\omega p)$, $\overset{2}{\Omega}$ for $\sqrt{T}P(c\omega q)$, $\overset{3}{\Omega}$ for $\sqrt{T}P(c, \omega, -r)$. Then the equation of this article is

$$\Omega + \overset{2}{\Omega} + x = \xi \text{ or } \cot\{\Omega + \overset{0}{\Omega} + x\} = \cot\xi,$$

that is $$\cot\{\Omega + \overset{2}{\Omega} + x\} = \sin\psi\,.\,\sin\psi_0\,.\,\frac{b\cos\omega}{\cos\omega^0}.$$

5. *Conjugate Hypernomisci.* For Conjugate Paranomes we had in Art. 14 of Ch. x.,

$$\frac{1+p}{p}\,.\,\Pi(c\omega p) - \frac{1-r}{r}\,.\,\Pi(c\omega, -r) - \frac{c^2}{pr}\,.\,F(c\omega) = \int_0 \frac{dV}{1 + pr\,.\,V^2},$$

where $V = \sin\omega\,.\,\sin\omega^0$, and in Art. 23 we found that

$$\sqrt{(pr)}(p^{-1} + 1) = \sqrt{T}(p), \quad \sqrt{(pr)}(r^{-1} - 1) = \sqrt{T}(-r).$$

When $$\omega = \tfrac{1}{2}\pi, \ \frac{1+p}{p}\,.\,\Pi_c(p) - \frac{1-r}{r}\,.\,\Pi_c(-r) - \frac{c^2}{pr}\,.\,F_c = 0,$$

[for $\frac{1}{\sqrt{(pr)}}\,.\,\tan^{-1}V$ vanishes with V, $\therefore$ with ω^0.

Multiply the former by $\sqrt{pr}$, the latter by $\sqrt{(pr)} \cdot \frac{F}{F_c}$ and subtract:

then $\sqrt{TP}\,(c\omega p) - \sqrt{T} \cdot P\,(c\omega, -r) = \tan^{-1}(\sqrt{pr} \cdot \sin\omega \cdot \sin\omega^0)$,

or
$$\Omega - \overset{3}{\Omega} = \tan^{-1}(\sqrt{pr} \cdot \sin\omega \sin\omega^0),$$
$$\tan(\Omega - \overset{3}{\Omega}) = \frac{c\cos\psi}{\cos\psi_0} \cdot \sin\omega \cdot \sin\omega^0;$$

since
$$\sqrt{(pr)} = \cot\psi \cdot \Delta(b\psi) = \frac{c\cos\psi}{\cos\psi_0}.$$

The comparison of the right-hand member here with that of the preceding article, is very suggestive.

6. Between the results of the two last Articles we may eliminate Ω as follows.

Put
$$f^{-1} = \frac{\Delta(c\omega)}{\tan\omega} \cdot \frac{\sin\psi\cos\psi}{\Delta(b\psi)}; \quad g = \frac{\Delta(b\psi)}{\tan\psi} \cdot \frac{\sin\omega\cos\omega}{\Delta(c\omega)};$$
$$h = \frac{\Delta(b\psi)\sin\omega}{\Delta(c\omega)\sin\psi};$$

then
$$f = \frac{h}{\cos\omega\cos\psi}, \quad g = h\cos\omega\cos\psi;$$
$$\begin{cases} \Omega + \overset{2}{\Omega} + x = \xi = \tan^{-1} \cdot f, \\ \Omega - \overset{3}{\Omega} = \tan^{-1} \cdot g. \end{cases}$$

Hence
$$\overset{2}{\Omega} + \overset{3}{\Omega} + x = \tan^{-1} f - \tan^{-1} g = \tan^{-1} \cdot \frac{f-g}{1+fg}.$$

But $f - g = f(1 - \cos^2\omega\cos^2\psi)$; $1 + fg = 1 + h^2$
$$= \frac{\Delta^2(c\omega)\sin^2\psi + \Delta^2(b\psi)\sin^2\omega}{\Delta^2(c\omega) \cdot \sin^2\psi} = \frac{\sin^2\omega + \sin^2\psi - \sin^2\omega\sin^2\psi}{\Delta^2(c\omega) \cdot \sin^2\psi}.$$

Here the numerator
$$= 1 - (1 - \sin^2\omega)(1 - \sin^2\psi) = 1 - \cos^2\omega\cos^2\psi,$$
which cancels with $f - g$,
$$\therefore \frac{f-g}{1+fg} = f \cdot \Delta^2(c\omega)\sin^2\psi = \frac{\Delta(c\omega)}{\cot\omega} \cdot \frac{\Delta(b\psi)}{\cot\psi}.$$

Finally
$$\tan(\overset{2}{\Omega} + \overset{3}{\Omega} + x) = \frac{f-g}{1+fg} = \frac{\Delta(c\omega)}{\cot\omega} \cdot \frac{\Delta(b\psi)}{\cot\psi} = \frac{\sin\omega}{\sin\omega^0} \cdot \frac{\sin\psi}{\sin\psi_0}.$$

Thus out of $\Omega\overset{2}{\Omega}\overset{3}{\Omega}$ we may select the one most convenient to calculate, and from it deduce either of the others.

7. We found in Ch. x. Art. 17, that when $F\omega + F\theta = F\eta$, we have

$$\sqrt{T}(\Pi\omega + \Pi\theta - \Pi\eta) = \tan^{-1} . \frac{\sqrt{T} . p \sin\omega \sin\theta \sin\eta}{1 + p(1 - \cos\omega\cos\theta\cos\eta)},$$

$$\therefore \sqrt{T}(\Pi\omega + \Pi\omega^0 - \Pi_c) = \tan^{-1} . \left\{\frac{\sqrt{T}}{1+p} . p \sin\omega \sin\omega^0\right\}.$$

Let $\eta = \frac{1}{2}\pi,\ \cos\eta = 0,\ \sin\eta = 1,\ \theta = \omega^0.$

Also $\sqrt{T} . \dfrac{p}{1+p} = \dfrac{\Delta(b\psi)}{\tan\psi} = \dfrac{c\cos\psi}{\cos\psi_0}$:

Therefore the last on the right

$$= \tan^{-1}\left\{\frac{c\cos\psi}{\cos\psi_0} . \sin\omega \sin\omega^0\right\},$$

which in Art. 5 we found to be $\Omega - \overset{3}{\Omega}$.

Since $F\omega + F\omega^0 - F_c = 0,\ \therefore (F\omega + F\omega^0 - F_c)\dfrac{\Pi_c}{F_c} = 0.$

Subtract this from the left-hand member of the preceding; then, observing that $\Pi(\omega) - F(\omega)\dfrac{\Pi_c}{F_c} = P(\omega),$

you find $\sqrt{T} . P(c\omega p) + \sqrt{T}P(c\omega^0 p) = \Omega - \overset{3}{\Omega}.$

But Ω is nothing but $\sqrt{T}P(c\omega p)$, therefore $\overset{3}{\Omega} = -\sqrt{T}P(c\omega^0 p)$, and the elements of $\overset{3}{\Omega}$ are $c\omega$ and $-r$. Thus we may change the negative parameter $-r$ into p its conjugate, if at the same time we change ω into $-\omega^0$. Or conversely.

8. From the general value of

$$\sqrt{T}(\Pi\omega + \Pi\theta - \Pi\eta),\ \text{when}\ F\omega + F\theta = F\eta,$$

if you subtract $\sqrt{T} . (F\omega + F\theta - F\eta) . \dfrac{\Pi_c}{F_c} = \text{zero},$

and if also ΘH are related to $\theta\eta$ as Ω to ω, you deduce

$$\sqrt{T}(P\omega + P\theta - P\omega) = \sqrt{T}(\Pi\omega + \Pi\theta - \Pi\eta),$$

an arc of which we know the tangent,

$$\therefore \tan(\Omega + \Theta - H) = \frac{\sqrt{T} . p \sin\omega \sin\theta \sin\eta}{1 + p(1 - \cos\omega\cos\theta\cos\eta)}.$$

9. *Commutation of Logarithmic Hypernomisci.* Recur to the case of $p = -c^2 \sin^2\theta$ and deduce the Hypernomiscus. We have

$$\sqrt{-T.} = \cot\theta . \Delta(c\theta),$$

whence

$$\sqrt{-T}\{\Pi(c\omega p) - F(c\omega)\} = G(c\theta) . F(c\omega) - \tfrac{1}{2}\{\daleth(c\eta) - \daleth(c\kappa)\},$$

if
$$\begin{Bmatrix} F\omega + F\theta = F\eta \\ F\omega - F\theta = F\kappa \end{Bmatrix}.$$

Make $\qquad \omega = \tfrac{1}{2}\pi, \ \therefore \ \sqrt{-T}\{\Pi_c(p) - F_c\} = G(c\theta) . F_c,$

as in Cor. to Art. 7, Ch. x.

Multiply the last by $\dfrac{F(c\omega)}{F_c}$, and subtract from the penultimate,

$$\therefore \ \sqrt{-T} . P(c\omega p) = \tfrac{1}{2}\{\daleth(c\kappa) - \daleth(c\eta)\}.$$

To exchange ω and θ leaves η unchanged, and does but change κ into $-\kappa$, without any effect upon $\daleth(c\kappa)$, since $\daleth$ is an even function. Therefore $\sqrt{-T} . P(c\omega p)$ remains unchanged, if we *commute* ω and θ, i.e., if we take θ for the amplitude and $-c^2 \sin^2\omega$ for the parameter. This will presently suggest Commutation in the case of circular parameters.

10. The last Article may also be exhibited in series of the Mesonomes, since we have Ch. VIII. Art. 8,

$$\daleth(c\omega) = \frac{1 - \cos 2x}{\operatorname{Sin} 2\rho} + \frac{1 - \cos 4x}{2 \operatorname{Sin} 4\rho} + \frac{1 - \cos 6x}{3 \operatorname{Sin} 6\rho} + \&c.$$

Write η and κ separately in this formula for ω (c and ρ remaining unchanged); then the Mesonome x must be changed to $x + t$ for amplitude η and to $x - t$ for amplitude κ, if t is the Mesonome to amplitude θ. Hence

$$-\sqrt{-T} . P(c\omega p) = \tfrac{1}{2}\left\{\frac{\cos(2x - 2t) - \cos(2x + 2t)}{\operatorname{Sin} 2\rho} + \frac{\cos(4x - 4t) - \cos(4x + 4t)}{2 \operatorname{Sin} 4\rho} + \&c.\right\}$$

$$= \frac{\sin 2x \sin 2t}{\operatorname{Sin} 2\rho} + \frac{\sin 4x \sin 4t}{2 \operatorname{Sin} 4\rho} + \frac{\sin 6x \sin 6t}{3 \operatorname{Sin} 6\rho} + \&c.$$

(given by Gudermann first). This result is visibly unaltered by exchanging x and t, that is, by commuting ω with θ.

11. Our first thought may be, to apply this series to the case of a circular parameter. But if we assume $\sin\theta = \sqrt{-1}\tan\psi$, which yields (from Mesonome y) $Ct = \sqrt{-1}\,By$, the factor $\sin 2nt$ changes to an Anticyclic, with great damage to the convergence. Yet let us examine it. We have now

$$\sqrt{T:P}\,.\,(c\omega p) = \Sigma\,\frac{\sin 2nx}{n\,\mathrm{Sin}\,2n\rho}\,.\,\mathrm{Sin}\left(\frac{4n\rho y}{\pi}\right),$$

which converges if $\dfrac{2y}{\pi}$ is <1 or $y < \frac{1}{2}\pi$, as indeed is implied in the parameter $c^2\tan^2\psi$ being finite, with y Mesonome to ψ. Nor is this all: but from Art. 4, Ω and $\overset{2}{\Omega}$ can be found from one another; hence of p and q we may choose the smaller. If p is $>c$, when $p = c^2\tan^2\psi$, then q is $<c$, where $q = \cot^2\psi = c^2\tan^2\psi_0$. By passing from p to q, we pass from ψ to ψ_0, from $F(b\psi) = By$ to $F(b\psi_0) = By_0$, and yy_0 being Conjugates, you have $y + y_0 = \frac{1}{2}\pi$, therefore of y and y_0 one is *less* than $\frac{1}{4}\pi$.

Thus, by rightly selecting p less than q, the *worst* case of convergence, incurred in

$$\Sigma\,\frac{\sin 2nx}{n\,\mathrm{Sin}\,2n\rho}\,.\,\mathrm{Sin}\,.\left(\frac{4n\rho y}{\pi}\right),$$

is the extreme case of $y = \frac{1}{4}\pi$, which makes the series

$$\Sigma\,\frac{\sin 2nx}{n\,\mathrm{Sin}\,.\,2n\rho}\,.\,\mathrm{Sin}\,(n\rho).$$

But
$$\mathrm{Sin}\,2n\rho = 2\,\mathrm{Sin}\,n\rho\,.\,\mathrm{Cos}\,n\rho.$$

Hence our worst case is

$$\Sigma\,\frac{\sin 2nx}{2n\,.\,\mathrm{Cos}\,n\rho} \text{ or } \frac{\sin 2x}{2\,\mathrm{Cos}\,\rho} + \frac{\sin 4x}{4\,\mathrm{Cos}\,2\rho} + \frac{\sin 6x}{6\,\mathrm{Cos}\,3\rho} + \&c.$$

Thus, to our surprize, Gudermann's formula in Art. 10 solves the case of the circular parameter also, if all is to turn on the convergence of the series.

Perhaps it will now be found that Gudermann's elaborate tables of $\log\mathrm{Sin}\,x$ and $\log\mathrm{Cos}\,x$, though only to eight decimals, do really give us mastery over

$$\Sigma\,\frac{\sin 2nx}{n\,\mathrm{Sin}\,2n\rho}\,.\,\mathrm{Sin}\,.\,\frac{4n\rho y}{\pi}.$$

But in every case, the *trouble of finding the Mesonome y* from a given ψ, according to received methods, may be as vexatious as our main problem.

[COR. In this calculation we need to find y as well as x, from $F(b\psi)$ as well as $F(c\dot{\omega})$, but the successive $b_1 b_2 b_3 \ldots$ needed to find y follow the law of Lander and not that of Gauss. This seems to give advantage over Arts. 13, 14 below.]

12. To deal with circular parameters, we may also apply Lagrange's scale to the Hypernomiscus. Of reciprocal parameters p and q (which make $pq = c^2$), one must be less than c. Let p be $< c$, also $c < b$. Let $c_1\omega_1 c_2\omega_2 \ldots$ follow the law of Lagrange's scale. Also let $p_1 r$ be disposable constants. Write Π_1 for $\Pi(c_1\omega_1 p_1)$ and T_1 for $T(p_1)$. Then

$$\Pi_1 = \int_0 \frac{dF_1}{1 + p_1 \sin^2\omega_1} = \int_0 \frac{(1+b)\,dF}{1 + p_1 \,.\, [(1+b)^2 \sin^2\omega \,.\, \sin^2\omega^0]}$$

$$= \int \frac{\Delta^2(c\omega)\ \ (1+b)\,dF}{\Delta^2(c\omega) + p_1(1+b)^2 \sin^2\omega\cos^2\omega}.$$

For from $F = \frac{1}{2}(1 + c_1)F_1$ we have $F_1 = (1+b)F$.

If now $v = \sin\omega$, our denominator is of the form $1 \pm \alpha v^2 - \beta v^4$. Take p_1 and r so as to make this denominator to be $(1 + pv^2)(1 - rv^2)$. First put

$$\omega = \tfrac{1}{2}\pi,\ v = 1,\ \therefore\ (1+p)(1-r) = 1 - c^2 = b^2,$$

then $-r$ is conjugate to p. Next, equate coefficients of v^4,

$$\therefore\ pr = p_1(1+b)^2, \text{ or } p_1 = \frac{pr}{(1+b)^2}.$$

[Remember that $p - r = pr - c^2$, by law of conjugates.]

Make q_1 reciprocal to p_1, that is,

$$p_1 q_1 = c_1^{\,2} = \left(\frac{1-b}{1+b}\right)^2.$$

Then from $\quad p_1 = \dfrac{pr}{(1+b)^2}$ you have $q_1 = \dfrac{(1-b)^2}{pr}$,

whence $\quad T_1 = (1+p_1)(1+q_1) = \dfrac{[(1+b)^2 + pr][(1-b)^2 + pr]}{(1+b)^2 \,.\, pr}$,

of which the numerator $= c^4 + (4 - 2c^2)\,pr + p^2r^2$,

or $\quad (pr - c^2)^2 + 4pr$ or $(p-r)^2 + 4pr$ or $(p+r)^2$.

Hence $$\sqrt{T_1} = \frac{p+r}{(1+b)\sqrt{(pr)}}.$$

We have now $$\Pi_1 = \int_0 \frac{(1+b)(1-c^2v^2)}{(1+pv^2)(1-rv^2)} \, . \, dF.$$

Let $$\frac{1-c^2v^2}{(1+pv^2)(1-rv^2)} = \frac{L}{1+pv^2} + \frac{M}{1-rv^2};$$

multiply by $1+pv^2$, and make $v^2 = -p^{-1}$,

$$\therefore \; L = \frac{1+c^2p^{-1}}{1+rp^{-1}} = \frac{p+c^2}{p+r}.$$

But $p + c^2 = \overline{p+1} \, . \, r$,

$$\therefore \; L = \frac{(p+1)r}{p+r} = \frac{pr}{p+r} \, . \, (1+p^{-1}).$$

So $$M = \frac{-pr}{p+r}(1-r^{-1}).$$

Thus $$\Pi_1 = \int_0 (1+b) \frac{pr}{p+r} \left\{ \frac{1+p^{-1}}{1+pv^2} + \frac{r^{-1}-1}{1-rv^2} \right\} dF.$$

Multiply by $$\sqrt{T_1} = \frac{p+r}{(1+b)\sqrt{(pr)}},$$

$$\therefore \; \sqrt{T_1}\Pi_1 = \int_0 \sqrt{(pr)} \left\{ \frac{p^{-1}+1}{1+pv^2} + \frac{r^{-1}-1}{1-rv^2} \right\} dF.$$

But and
$$\left. \begin{array}{l} \sqrt{T(p)} = \sqrt{(pr)}\,(p^{-1}+1) \\ \sqrt{T(-r)} = \sqrt{(pr)}\,(r^{-1}-1) \end{array} \right\}, \text{ Art. 23 of Ch. X.}$$

$$\therefore \; \sqrt{T_1}\Pi_1 = \sqrt{T}\Pi\,(c\omega p) + \sqrt{T}\Pi\,(c, \omega, -r),$$

a very elegant property.

13. Pass from Hypernome to Hypernomiscus. Make $\omega = \frac{1}{2}\pi$, $\omega_1 = \pi$ according to Lagrange's scale,

$$\therefore \; 2\sqrt{T_1}\Pi\,(c_1p_1) = \sqrt{T}\Pi\,(cp) + \sqrt{T}\Pi\,(c, -r).$$

Multiply by $$\frac{1}{2}\frac{F(c_1\omega_1)}{F(c_1 \frac{1}{2}\pi)} = \frac{F(c\omega)}{F(c_1 \frac{1}{2}\pi)}$$

and subtract from the Hypernomes of last Article,

$$\therefore \; \sqrt{T_1}P_1 = \sqrt{T}P\,(c\omega p) + \sqrt{T}P\,(c\omega - r),$$

or $$\Omega + \Omega_3 = \Omega_1, \;\; \Omega - \Omega_3 = \tan^{-1}\{\sqrt{(pr)} \sin\omega \, . \, \sin\omega^0\}.$$

With this combine from Art. 5, and observe that

$$\sqrt{(pr)} = \sqrt{p_1} \, . \, (1+b),$$

$$\therefore \; \sqrt{(pr)} \sin\omega \sin\omega^0 = \sqrt{p_1} \, . \, (1+b) \sin\omega \sin\omega^0 = \sqrt{p_1} \, . \, \sin\omega_1.$$

Eliminate Ω, $\therefore\ \Omega - \frac{1}{2}\Omega_2 = \frac{1}{2}\tan^{-1}(\sqrt{p_1}\,.\,\sin\omega_1)$,

the equation of reduction. Repeating it n times, you get

$$\Omega - 2^{-n}\Omega_n = \tfrac{1}{2}\tan^{-1}(\sqrt{p_1}\sin\omega_1) + \tfrac{1}{4}\tan^{-1}(\sqrt{p_2}\sin\omega_2) + \dots$$
$$+ 2^{-n}\,.\,\tan^{-1}(\sqrt{p_n}\,.\,\sin\omega_n) + \dots.$$

Make $n=\infty$, $c_n=0$, p_n is $< c_n$ and vanishes also. Then by Art. 4 Ω_n vanishes, and much more does $2^{-n}\Omega_n$,

$$\therefore\ \Omega = \Sigma\,.\,2^{-n}\tan^{-1}(\sqrt{p_n}\sin\omega_n),\ \text{where } n=1, 2, 3, 4\dots.$$

The remaining trouble is, to calculate the series $pp_1p_2p_3\dots$.

14. Legendre has gone before us herein. Let $p_nq_n = c_n^2$. We have

$$p_1 = \frac{pr}{(1+b)^2} = \frac{p}{(1+b)^2}\cdot\frac{p+c^2}{1+p} = \frac{p^2}{(1+b)^2}\cdot\frac{1+q}{1+p}.$$

Hence

$$q_1 = \frac{c_1^2}{p_1} = \left(\frac{1-b}{1+b}\right)^2\cdot\frac{1}{p_1} = \frac{(1-b)^2}{p^2}\cdot\frac{1+p}{1+q} = \frac{q^2}{(1+b)^2}\cdot\frac{1+p}{1+q}.$$

Thus, to exchange, p and q exchanges p_1 and q_1; also q_1 is deduced from q by the same law as p_1 from p.

Indeed if $p=\cot^2\psi$ and $p_1=\cot^2\psi_1$, the relation of p_1 to p yields

$$\cot\psi_1 = \frac{\cot\psi\,.\,\Delta(b\psi)}{1+b},$$

as in the scale of Gauss, in which the successive conjugates to the amplitudes are mutually related by the very same law as are the successive amplitudes among themselves.

In that scale, if $\sin\lambda_n = b_n\sin\psi_n$, we infer

$$\sin\lambda_{n-1} = \sqrt{b_{n-1}}\tan\tfrac{1}{2}\lambda_n.$$

Legendre's rule then is: Having taken ψ such that $p=\cot^2\psi$, find λ from the equation $\sin\lambda = b\sin\psi$; then from λ derive $\lambda_1\lambda_2\dots\lambda_n$ in succession. From $\lambda_n\lambda_{n-1}\dots\lambda_2\lambda_1$ deduce $\psi_n\psi_{n-1}\dots\psi_2\psi_1$ by the law $b_r\sin\psi_r = \sin\lambda_r$. Finally you attain $p_1p_2\dots p_n$ by the relation

$$p_r = \cot^2\psi_r.$$

15. Let $p_n = l_n\,.\,c_n$, then from

$$p_1 = \frac{p}{(1+b)^2}\cdot\frac{p+c^2}{1+p}$$

we have

$$l_1c_1 = \frac{lc}{(1+b)^2}\cdot\frac{lc+c^2}{1+lc};$$

but $$c_1 = \frac{c^2}{(1+b)^2}, \therefore l_1 = l \cdot \frac{l+c}{1+lc},$$

or $$1 \pm \frac{1}{l} = 1 \pm \frac{l+c}{1+lc} = \frac{(1 \pm l)(1 \pm c)}{1+lc}.$$

If then $l^2 < 1$, $1 \pm l$ being positive, so is $1 \pm \frac{l_1}{l}$ and l_1 is $< l$. Hence $l\, l_1 l_2 l_3 \ldots$ is a decreasing series. Therefore $p\, p_1 p_2 p_3 \ldots$ decrease *more* rapidly than $c\, c_1 c_2 c_3 \ldots$.

16. Notwithstanding all that can justly be said of the rapid convergence of $\sqrt{p_r} . \sin \omega_r$, the labour of computing with any accuracy, first $c_1 c_2 c_3 c_4 \ldots$, next $\omega_1 \omega_2 \omega_3 \ldots$, next $\lambda\, \lambda_1 \lambda_2 \lambda_3 \ldots$, next $\psi_1 \psi_2 \psi_3 \ldots$, next $p_1 p_2 p_3 \ldots$ with a view to $\tan^{-1}(\sqrt{p_r} \sin \omega_r)$, must at best be formidable: and if (we have tables of Sin u or log Sin u) the question will be, whether the process of finding the Mesonomes to $c\omega$ and to $b\psi$ is more troublesome than to calculate the fivefold series of c_r, ω_r, λ_r, ψ_r and p_r.

17. *Variation of the Parameter.* Suppose c and ω to be constant, p to vary, then

$$\frac{d\,(p\Pi)}{dp} = \int_0 \frac{d}{dp}\left(\frac{p}{1+p \sin^2 \omega}\right) dF = \int_0 \frac{dF}{(1+p\sin^2\omega)^2}.$$

By Ch. x. Art. 5, if $Q = \alpha + \beta v^2 + \gamma v^4$, with v now for x, and r for m,

$$S_r = \int \frac{dv}{(1+pv^2)^r \sqrt{Q}},$$

we can reduce S_r. To adapt it to present service, we have only to make $r = 2$.

We have from that Art.

$$v . \sqrt{Q}\,(1+pv^2)^{r-1} = A . S_r + B . S_{r-1} + C . S_{r-2} + D . S_{r-3}.$$

We assume $Q = (1-v^2)(1-c^2v^2)$, and when $r = 2$, C vanishes, having $(2r-4)$ as a factor. $S_1 = \Pi$;

$$S_{-1} = \int_0 (1+pv^2)\, dF = F + \frac{p}{c^2}(F - E).$$

Now $$v\sqrt{Q}(1+pv^2)^{-1} = A . S_2 + BS_1 + DS_{-1},$$

$$\left[S_2 = \int \frac{dF}{(1+pv^2)^2} = \frac{d\,(p\Pi)}{dp_1},\ S_1 = \Pi\right]$$

$$\alpha = 1,\ -\beta = 1 + c^2,\ \gamma = c^2.$$

We had

$$A = (2r-2)(\alpha - \beta p^{-1} + \gamma p^{-2}) = (2r-2)(1+p^{-1})(1+c^2p^{-1}),$$
$$-B = (2r-3)(\alpha - 2\beta p^{-1} + 3\gamma p^{-2}).$$

Put $r = 2$, $\therefore\ A = 2Tp^{-1}$, also

$$A + B = \alpha - \gamma p^{-2} = 1 - c^2p^{-2}.$$

Now
$$T = (1+p)(1+c^2p^{-1}) = 1 + p + c^2p^{-1} + c^2,$$
$$\frac{dT}{dp} = 1 - c^2p^{-2}, \text{ or } A + B = \frac{dT}{dp}, \text{ with } Ap = 2T.$$

Again, $+D = (2r-5)\,.\,\gamma p^{-2}$, or $= -c^2p^{-2} = \dfrac{dT}{dp} - 1$, with $v = \sin\omega$.

Hence
$$\frac{v\sqrt{Q}}{1+pv^2} = A\,.\,S_2 + B\,.\,S_1 + D\,.\,S_{-1},$$

where
$$S_2 = \int_0 \frac{dF}{(1+pv^2)^2} = \frac{d(p\Pi)}{dp} = \Pi + p\frac{d\Pi}{dp},$$

$$\therefore\ \frac{v\sqrt{Q}}{1+pv^2} = A\left\{\Pi + p\frac{d\Pi}{dp}\right\} + B\,.\,\Pi - D\,.\,S_{-1}$$
$$= (A+B)\,.\,\Pi + Ap\,.\,\frac{d\Pi}{dp} - D\left(F + \frac{p}{c^2}\,.\,\overline{F-E}\right)$$
$$= \left(\frac{dT}{dp}\,.\,\Pi + 2T\,.\,\frac{d\Pi}{dp}\right) + \left(1 - \frac{dT}{dp}\right)\left(F + \frac{p}{c^2}\,.\,\overline{F-E}\right).$$

To make the two first terms an exact differential, multiply by $\dfrac{dp}{2\sqrt{T}}$, and then integrate: (differentiation verifies sufficiently:)

$$\tfrac{1}{2}v\sqrt{Q}\,.\int_0 \frac{dp}{(1+pv^2)\sqrt{T}}$$
$$= \sqrt{T}\,.\,\Pi + F\int_0 \frac{dp}{2\sqrt{T}} - F\sqrt{T} + (F-E)\int \frac{p}{c^2}\left(1 - \frac{dT}{dp}\right)\,.\,\frac{dp}{2\sqrt{T}}.$$

In the last integral, restore $\dfrac{c^2}{p^2}$ for its equivalent $1 - \dfrac{dT}{dp}$ and it becomes

$$(F-E)\int \frac{dp}{p\sqrt{T}},$$

so that the right-hand member becomes

$$\sqrt{T}\Pi + \tfrac{1}{2}F\,.\int_0 \frac{dp}{\sqrt{T}} - F\sqrt{T} + \tfrac{1}{2}(F-E)\int_0 \frac{dp}{p\sqrt{T}}.$$

Couple the first and third terms into $\sqrt{T}(\Pi - F)$ which vanishes with p. Thus we may begin the other integrals from $p = 0$, without any constant of integration that might contain c and ω.

18. This last integration by Legendre gives a direct method of attaining Π_c when T is positive, without passing through imaginary arcs, as in Ch. x. Arts. 22, 23; therefore the corroboration will interest us.

We must first integrate $\int_0 \frac{dp}{\sqrt{T}}$ and $\int_0 \frac{dp}{p\sqrt{T}}$.

Let
$$p = c^2 \tan^2\psi, \quad \sqrt{T}^{-1} = \sin\psi \,.\, \sin\psi_0,$$
$$dp = 2c^2 \tan\psi \,.\, d\tan\psi = 2c^2 \tan\psi \,.\, \sec^2\psi\, d\psi,$$

and
$$\sin\psi_0 = \frac{\cos\psi}{\Delta(b\psi)}.$$

$$\int_0 \frac{dp}{\sqrt{T}} = \int_0 2c^2 \,.\, \tan\psi \,.\, \sec^2\psi \,.\left(\sin\psi \,.\, \frac{\cos\psi}{\Delta(b\psi)}\right) . \,d\psi$$
$$= \int_0 2c^2 \tan^2\psi \,.\, \frac{d\psi}{\Delta(b\psi)} = 2\tan\psi \,.\, \Delta(b\psi) - 2E(b\psi)$$

by 8th formula in the Table of Ch. I. Art. 2,

and
$$\int_0 \frac{dp}{p\sqrt{T}} = \int_0 \frac{2d\psi}{\Delta(b\psi)} = 2F(b\psi).$$

Hence $\frac{1}{2} v\sqrt{Q} \,.\, \int_0 \frac{dp}{(1+pv^2)\sqrt{T}}$

$$= \sqrt{T} \,.\, (\Pi - F) + F\{\tan\psi \,.\, \Delta(b\psi) - E(b\psi)\} + (F - E) \,.\, F(b\psi).$$

Take first
$$\left.\begin{matrix} \omega = \frac{1}{2}\pi \\ Q = 0 \end{matrix}\right\}, \text{ and divide by } F,$$

$$\therefore\ \sqrt{T}\left\{1 - \frac{\Pi_c}{F_c}\right\} = \{\tan\psi \,.\, \Delta(b\psi) - E(b\psi)\} + (1 - \aleph_c) \,.\, F(b\psi)$$
$$= \tan\psi \,.\, \Delta(b\psi) - J(b\psi).$$

This agrees with Ch. x. Art. 22, though what there was called ψ_0 is here ψ.

Multiply the last by $F(c\omega)$ and subtract the result from the previous equation of Art. 17 in which ω was indefinite. This process eliminates

$$\tan\psi \,.\, \Delta(b\psi) - E(b\psi); \text{ and } F(c\omega) \,.\, F(b \,.\, \psi).$$

Observe that
$$E(c\omega) - \aleph_c F(c\omega) = G(c\omega).$$

19. Thus we obtain

$$v\sqrt{Q}\int_0 \frac{\frac{1}{2}dp}{(1+pv^2)\sqrt{T}} = \sqrt{T}\left(\Pi - \Pi_c . \frac{F}{F_c}\right) - \left(E - \frac{E_c}{F_c} . F\right) . F(b, \psi)$$
$$= \sqrt{T} . P . (c\omega p) - G(c_1\omega) . F(b\psi)$$

and it remains to reduce the left-hand member, in which

$$v\sqrt{Q} = \sin\omega\cos\omega\,\Delta(c\omega).$$

Observe that

$$\frac{\frac{1}{2}dp}{(1+pv^2)\sqrt{T}} = \frac{p}{1+pv^2} . \frac{\frac{1}{2}dp}{p\sqrt{T}} = \frac{p}{1+pv^2} . dF(b\psi).$$

Also $$\frac{p}{1+v^2p} = \frac{c^2\tan^2\psi}{1+\sin^2\omega . c^2\tan^2\psi} = \frac{c^2\sin^2\psi}{\cos^2\psi + c^2\sin^2\omega . \sin^2\psi}$$
$$= \frac{c^2\sin^2\psi}{1-\Delta^2(c\omega) . \sin^2\psi},$$

$$\therefore\ v\sqrt{Q} . \int_0 \frac{\frac{1}{2}dp}{(1+pv^2)\sqrt{T}} = \frac{c^2\sin\omega\cos\omega}{\Delta(c\omega)} . \int \frac{\Delta^2(c\omega) . \sin^2\psi}{1-\Delta^2(c\omega)\sin^2\psi} . dF(b\psi).$$

Again, $$\frac{\Delta^2(c\omega) . \sin^2\psi}{1-\Delta^2(c\omega) . \sin^2\psi} = \left\{\frac{1}{1-\Delta^2(c\omega) . \sin^2\psi} - 1\right\}.$$

Hence if $+r' = -\Delta^2(c\omega)$ a negative parameter to modulus b,

$$\int_0 \frac{dF(b\psi)}{1-r' . \sin^2\psi} = \Pi\,(b\psi r').$$

Thus the new equation is

$$\frac{c^2\sin\omega\cos\omega}{\Delta(c\omega)} . \{\Pi(b\psi r') - F(b\psi)\} = \sqrt{T} . P(c\omega p) - G(c\omega) . F(b\psi).$$

Each side vanishes when $p = 0$, $\psi = 0$; therefore no constant of integration is needed.

20. Further $T(+r') = (1+r')(1+b^2r'^{-1})$

$$= c^2\sin^2\omega . \left(1 - \frac{b^2}{\Delta^2(c\omega)}\right) = c^2 . \sin^2\omega . \frac{c^2\cos^2\omega}{\Delta^2(c\omega)},$$

whence $$\sqrt{T}(r') = \frac{c^2\sin\omega\cos\omega}{\Delta(c\omega)}.$$

Thus, distinguishing $T(r')$ from $T(p)$, we have

$$\sqrt{T(r')}\,\{\Pi(b\psi r') - F(b\psi)\} = \sqrt{T}(p) . P(c\omega p) - G(c\omega) . F(b\psi).$$

Make $\psi = \frac{1}{2}\pi$, or $p = \infty$. Then by Art. 3 above

$$\surd T(p) \,.\, P(c\omega p) = \tfrac{1}{2}\pi\left(1 - \frac{F(c\omega)}{F_c}\right),$$

$$\therefore\ \surd T(r')\,\{\Pi_b(r') - F_b\} = \tfrac{1}{2}\pi\left\{1 - \frac{F(c\omega)}{F_c}\right\} - G(c\omega) \,.\, F_b.$$

Multiply this by $\dfrac{F(b\psi)}{F_b}$, and subtract the result from the penultima.

Then $\quad \surd T(r') \,.\, \left\{\Pi(b\psi r') - \Pi_b(r') \,.\, \dfrac{F(b\psi)}{F_b}\right\}$

$$= \surd T(p) \,.\, P(c\omega p) - \tfrac{1}{2}\pi\left\{1 - \frac{F(c\omega)}{F_c}\right\} \,.\, \frac{F(b\psi)}{F_b}.$$

Here the left hand is $\surd T(r') \,.\, P(b\psi r')$.

21. If then we write $\Omega\overset{2}{\Omega}\overset{3}{\Omega}$ as in Arts. 4 and 5, related to elements $c\omega\psi$, the three parameters being

$$p = \cot^2\psi, \quad q = c^2\tan^2\psi, \quad -r = \Delta^2(b\psi),$$

we may suppose that the change of $c\omega\psi$ to $b\psi\omega$ changes $\Omega\overset{2}{\Omega}\overset{3}{\Omega}$ to $\Psi\overset{2}{\Psi}\overset{3}{\Psi}$. This may be called *Commutation*. [Only in *circular* Parameters c changes to b.] In the last Articles we had $p = c^2\tan^2\psi$, hence the $\surd T \,.\, P(c\omega p)$ of our last equation must be interpreted as $\overset{2}{\Omega}$. Further we have x, y as Mesonomes to $F(c\omega)$, $F(b\psi)$. Hence our last equation may stand as

$$\tfrac{1}{2}\pi \,.\, (\overset{2}{\Omega} - \overset{3}{\Psi}) = (\tfrac{1}{2}\pi - x) \,.\, y.$$

Only algebraic combinations are now needed.

22. In Art. 5, $\tan(\Omega - \overset{3}{\Omega}) = \dfrac{c\cos\psi}{\cos\psi_0} \,.\, \sin\omega\sin\omega^0$.

Commute: then

$$\tan(\Psi - \overset{3}{\Psi}) = \frac{b\cos\omega}{\cos\omega^0} \,.\, \sin\psi\sin\psi_0.$$

But here the right hand, by Art. 4,

$$= \cot(\Omega + \overset{2}{\Omega} + x), \quad \therefore\ \tfrac{1}{2}\pi - (\Psi - \overset{3}{\Psi}) = \Omega + \overset{2}{\Omega} + x,$$

or

$$(\tfrac{1}{2}\pi - x) - (\Omega + \Psi) = \overset{2}{\Omega} - \overset{3}{\Psi}.$$

From this and the result of Art. 21 eliminate $\overset{2}{\Omega} - \overset{3}{\Psi}$; then

$$\tfrac{1}{2}\pi(\Omega + \Psi) = (\tfrac{1}{2}\pi - x)(\tfrac{1}{2}\pi - y),$$

which is symmetrical as between the leading integrals.

23. Again, from

$$\tan(x + \overset{2}{\Omega} + \overset{3}{\Omega}) = \frac{\sin\omega}{\sin\omega^0} \cdot \frac{\sin\psi}{\sin\psi_0},$$

by Commutation you deduce

$$\tan(y + \overset{2}{\Psi} + \overset{3}{\Psi}) = \frac{\sin\psi}{\sin\psi_0} \cdot \frac{\sin\omega}{\sin\omega^0},$$

$$\therefore\ x + \overset{2}{\Omega} + \overset{3}{\Omega} = y + \overset{2}{\Psi} + \overset{3}{\Psi},$$

which does but give

$$\tfrac{1}{2}\pi(\overset{2}{\Psi} - \overset{3}{\Omega}) = (\tfrac{1}{2}\pi - y)\,x.$$

The great practical fact is, that when c is so large as to make convergence slow, *we need no reverse methods* as with F, G and ℸ, but by Commutation we can pass at once to a Ψ which will give us the advantage of a very small modulus b_1 and then from this Ψ we can deduce our Ω.

Any Hypernomiscus is thus in every case calculable: then from it by aid of the *complete* Hypernome any indefinite Hypernome or Paranome is in our power. Thus, in demonstrating our ability (when occasion requires) to evaluate these integrals, the treatise may seem to have reached its goal. But with three independent elements, even a single calculation is necessarily very laborious.

It would appear that the entire theory of the Paranome is due to the genius of Legendre.

END.

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